

Martingale Expansion in Mixed Normal Limit *

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Summary The quasi-likelihood estimator and the Bayesian type estimator of the volatility parameter are in general asymptotically mixed normal. In case the limit is normal, the asymptotic expansion was derived in [28] as an application of the martingale expansion. The expansion for the asymptotically mixed normal distribution is then indispensable to develop the higher-order approximation and inference for the volatility. The classical approaches in limit theorems, where the limit is a process with independent increments or a simple mixture, do not work. We present asymptotic expansion of a martingale with asymptotically mixed normal distribution. The expansion formula is expressed by the adjoint of a random symbol with coefficients described by the Malliavin calculus, differently from the standard invariance principle. Applications to a quadratic form of a diffusion process (“realized volatility”) are discussed.

Keywords and phrases Asymptotic expansion, martingale, mixed normal distribution, Malliavin calculus, random symbol, double Itô integral, quadratic form.

1 Introduction

The asymptotic expansion is a tool to give a precise approximation to the probability distribution. As commonly well known, it is the basement of the contemporary fields in theoretical statistics such as the asymptotic decision theory, prediction, information criterion, bootstrap and resampling methods, information geometry, and stochastic numerical analysis with applications to finance, as well as the higher-order approximation of distributions. The methodology of the asymptotic expansion has been well established, and historically well developed especially for independent observations ([2]).

For stochastic processes, there are two principles of asymptotic expansion. The first one is the mixing approach. It is also called the *local approach* because it takes advantage of the factorization of the characteristic function by the Markovian property more or less, and corresponds to the classical asymptotic expansion for independent models. A compilation of the studies for Markovian or near Markovian chains is Götze and Hipp [5]. The local nondegeneracy, that is, the decay of each factor of the characteristic function becomes an essential

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problem, and Götze and Hipp [6] showed it for time series models. For continuous time, Kusuoka and Yoshida [12] and Yoshida [30] treated the ϵ -Markovian process to give asymptotic expansion under the mixing condition. Then the local nondegeneracy of the Malliavin covariance works for objects expressed by the stochastic analysis. Though this method is relatively new, there are already not a few applications: expansion for a functional of the ϵ -Markov processes, statistical estimators, information criterion, stochastic volatility model, empirical distribution ([20], [24], [14], [13] among others).

Although the local approach is more efficient when one treats mixing processes, the asymptotic expansion for the ergodic diffusion was first derived through the martingale expansion ([28], [19], [29]). When the strong mixing coefficient decays sufficiently fast, under suitable moments conditions, the functionals appearing as the higher-order terms in the stochastic expansion of the functional bear a jointly normal limit distribution and it yields a classical expansion formula each of whose terms is the Hermite polynomial times the normal density. On the other hand, if we consider the sum of quadratics of the increments of a diffusion process, it is observed that in the higher-order, the variables have a non-Gaussian limit even when the sum is asymptotically normal in the first order. Such phenomenon is observed in the estimation of the statistical parameter in the essentially linearly parametrized diffusion coefficient. Due to the non-Gaussianity, we cannot apply the mixing approach in this case. Nonetheless, the martingale expansion can apply to obtain the expansion for the estimator of the volatility parameter; see [28]. This example shows that the martingale expansion is not inferior to the mixing method but even superior in some situations.

Estimation for the diffusion coefficient has been attracting statisticians' interests. There are many studies in theoretical statistics such as [3], [17], [18], [27], [11], [22], [8], [23], [21] among others. The estimators, including the so-called the realized volatility, are in general asymptotically mixed normal; see for example [3], [4], [7], [25]. Today vast literature about this topic is available around financial data analysis. We refer the reader to [1] and [16] and references therein for recent advances in the first-order asymptotic theory and access to related papers. The statistical theory for mixed limits is called the *non-ergodic* statistics since the Fisher information and the observed information are random even in the limit, differently from the classical cases where those are a constant. In this sense, it may be said that the mixing approach belongs to the classical theory. As for the asymptotic expansion in non-ergodic statistics, it seems that there is room for study.

In order to explain the technical difficulties in this question, let us recall the method in the proof of the martingale central limit theorem and the classical martingale expansion. Let $M^n = (M_t^n)_{t \in [0,1]}$ be a continuous martingale; it is possible to consider more general local martingale but the existence of jumps for example does not change the situation essentially. Under the condition that $\langle M^n \rangle_1 \rightarrow^p 1$ as $n \rightarrow \infty$, it suffices to show that

$$E[e^{iuM_1^n}]e^{\frac{1}{2}u^2} = E[e^{iuM_1^n + \frac{1}{2}u^2}] \rightarrow 1$$

as $n \rightarrow \infty$ for every $u \in \mathbb{R}$. Since for large n ,

$$\begin{aligned} E[e^{iuM_1^n + \frac{1}{2}u^2}] &= E[e^{iuM_1^n + \frac{1}{2}u^2 \langle M^n \rangle_1} e^{-\frac{1}{2}u^2(\langle M^n \rangle_1 - 1)}] \\ &\sim E[e^{iuM_1^n + \frac{1}{2}u^2 \langle M^n \rangle_1}] \\ &= 1 \end{aligned}$$

by the martingale property of the exponential functional if necessary by suitable localization. Thus we obtain the central limit theorem. Roughly speaking, if evaluating the gap $\langle M^n \rangle_1 - 1$ more precisely, we can obtain the asymptotic expansion. In this standard proof in the classical theory of the limit theorems for semimartingales converging to a process with independent increments, the commutativity of the expectation and $\mathcal{E}(M_1^\infty)(u)^{-1}$, where $\mathcal{E}(M_1^\infty)(u)$ is the characteristic function of the limit of M_1^n , was essential. However, when it is random, this method does not work. It explains the difficulty with the asymptotic expansion in the limit with random characteristics.

As already mentioned, the asymptotic expansion is inevitable to form modern theories in the non-ergodic statistics, and so it seems natural to try to extend the classical theory of asymptotic expansion in the ergodic statistics to a theory that is applicable to the non-ergodic statistics. The aim of this article is to present a new martingale expansion to answer this question.

As a prototype problem, in Section 8, for a diffusion process satisfying the Itô integral equation

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dw_s \quad (1)$$

we will consider a quadratic form of the increments of X with a strongly predictable kernel:

$$U_n = \sum_{j=1}^n c(X_{t_{j-1}})(\Delta_j X)^2, \quad (2)$$

where c is a function, $\Delta_j X = X_{t_j} - X_{t_{j-1}}$ and $t_j = j/n$. Under suitable conditions, U_n has the in-probability limit

$$U_\infty = \int_0^1 c(X_s)\sigma(X_s)^2 ds. \quad (3)$$

Then the problem is to derive second-order approximation of the distribution of $\zeta_n = \sqrt{n}(U_n - U_\infty)$. When $c \equiv 1$, this gives asymptotic expansion of the realized volatility. It turns out that ζ_n admits a stochastic expansion with a double stochastic integral perturbed by higher-order terms. In Section 6, we will present an expansion formula for a general perturbed double stochastic integral. When the function σ of (1) involves an unknown parameter θ and $b(X_t)$ is unobservable, this gives a semiparametric estimation problem. The error of the quasi maximum likelihood estimator $\hat{\theta}_n$ of θ has a representation by a perturbed double stochastic integral, the kernel of which is different from that of the realized volatility. Our result provides asymptotic expansion for $\hat{\theta}_n$ though we do not go into this question here. Access Section 6.3, Section 7.1 or Section 8 first if the reader wants to know quickly the results in typical problems of a quadratic form, while the results presented in the preceding sections are more general.

The organization of this paper is as follows. In Section 2, we define our object and prepare some notation. We introduce the notion of random symbols and define an adjoint operation of random symbols in Section 3. The second-order part in the asymptotic expansion consists of two terms and they are represented by certain random symbols. The first one (adaptive symbol) corresponds to the second-order correction term of the classical martingale expansion in [28], while the second one (anticipative symbol) is new. Since the characteristics of the targeted distribution are random, it is natural to consider symbols with randomness. An error bound of the asymptotic expansion will be presented in Section 4. At this stage, the expression of the second-order term associated with the anticipative random symbol is not explicit yet. In Section 6, we treat a variable whose principal part is given by a sum of double stochastic integrals. It is natural to consider such a variable because it appears in the context of the statistical inference for diffusion coefficients and realized volatility. It turns out that the anticipative random symbol involves objects from the infinite dimensional calculus. Section 7 gives expansion for a simple quadratic form of the increments of a Brownian motion. Studentization procedure is also discussed there. The arguments on the nondegeneracy will be applied in Section 8, where we derive the asymptotic expansion for a quadratic form of the increments of a diffusion process. Precise approximation to the realized volatility in finance is one of applications of our result though our aim is to develop a new methodology in the theory of limit theorems.

2 Functionals

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, P)$ be a stochastic basis with $\mathcal{F} = \mathcal{F}_1$. On Ω , we will consider a sequence of d -dimensional functionals each of which admits the decomposition

$$Z_n = M_n + W_n + r_n N_n.$$

For every $n \in \mathbb{N}$, $M^n = (M_t^n)_{t \in [0,1]}$ denotes a d -dimensional martingale with respect to \mathbf{F} and M_n denotes the terminal variable of M^n , i.e., $M_n = M_1^n$. In the above decomposition, $W_n, N_n \in \mathcal{F}(\Omega; \mathbb{R}^d)^1$ and $(r_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers tending to zero as $n \rightarrow \infty$.

We will essentially treat a conditional expectation given a certain functional $F_n \in \mathcal{F}(\Omega; \mathbb{R}^{d_1})$, $n \in \mathbb{N}$. Later we will specify those functionals more precisely to validate computations involving conditional expectations.

We write

$$M_\infty = M_1^\infty, \quad C_t^n = \langle M^n \rangle_t, \quad C_n = \langle M^n \rangle_1.$$

¹The set of d -dimensional measurable mappings.

Process M^∞ will be specified later. Let $C_\infty \in \mathcal{F}(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d)$, $W_\infty \in \mathcal{F}(\Omega; \mathbb{R}^d)$ and $F_\infty \in \mathcal{F}(\Omega; \mathbb{R}^{d_1})$. The tangent random vectors are given by

$$\begin{aligned}\overset{\circ}{C}_n &= r_n^{-1}(C_n - C_\infty), \\ \overset{\circ}{W}_n &= r_n^{-1}(W_n - W_\infty), \\ \overset{\circ}{F}_n &= r_n^{-1}(F_n - F_\infty).\end{aligned}$$

Consider an extension

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \overset{\circ}{\Omega}, \mathcal{F} \times \overset{\circ}{\mathcal{F}}, P \times \overset{\circ}{P})$$

of (Ω, \mathcal{F}, P) by a probability space $(\overset{\circ}{\Omega}, \overset{\circ}{\mathcal{F}}, \overset{\circ}{P})$. Let $M^\infty \in \mathcal{F}(\bar{\Omega}; C([0, 1]; \mathbb{R}^d))$, $N_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d)$, $\overset{\circ}{C}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d \otimes \mathbb{R}^d)$, $\overset{\circ}{W}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d)$ and $\overset{\circ}{F}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^{d_1})$.

We assume ²

- [B1]** (i) $(M^n, N_n, \overset{\circ}{C}_n, \overset{\circ}{W}_n, \overset{\circ}{F}_n) \rightarrow^{d_s(\mathcal{F})} (M^\infty, N_\infty, \overset{\circ}{C}_\infty, \overset{\circ}{W}_\infty, \overset{\circ}{F}_\infty)$.
(ii) $\mathcal{L}\{M_t^\infty | \mathcal{F}\} = N_d(0, C_t^\infty)$.

The convergence in (i) is \mathcal{F} -stable convergence. In particular, $(C_n, W_n, F_n) \rightarrow^p (C_\infty, W_\infty, F_\infty)$, the variables C_∞ , W_∞ and F_∞ are \mathcal{F} -measurable, and

$$(M_n, N_n, \overset{\circ}{C}_n, \overset{\circ}{W}_n, \overset{\circ}{F}_n, C_n, W_n, F_n) \rightarrow^d (M_\infty, N_\infty, \overset{\circ}{C}_\infty, \overset{\circ}{W}_\infty, \overset{\circ}{F}_\infty, C_\infty, W_\infty, F_\infty).$$

Let $\check{\mathcal{F}} = \mathcal{F} \vee \sigma[M_1^\infty]$. Then there exists a measurable mapping $\check{C}_\infty : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\check{C}_\infty(\omega, M_\infty) = E[\overset{\circ}{C}_\infty | \check{\mathcal{F}}]$$

and we simply write it as $\check{C}_\infty(M_\infty)$. The uniqueness of the mapping \check{C}_∞ is not necessary in what follows. In the same way, we define $\check{W}_\infty(\omega, z)$, $\check{F}_\infty(\omega, z)$ and $\check{N}_\infty(\omega, z)$ so that

$$\begin{aligned}\check{W}_\infty(\omega, M_\infty) &= E[\overset{\circ}{W}_\infty | \check{\mathcal{F}}] \\ \check{F}_\infty(\omega, M_\infty) &= E[\overset{\circ}{F}_\infty | \check{\mathcal{F}}] \\ \check{N}_\infty(\omega, M_\infty) &= E[N_\infty | \check{\mathcal{F}}].\end{aligned}$$

Further, we introduce the notation

$$\tilde{C}_\infty(z) \equiv \check{C}_\infty(\omega, z) := \check{C}_\infty(\omega, z - W_\infty)$$

and similarly $\tilde{W}_\infty(\omega, z)$, $\tilde{F}_\infty(\omega, z)$ and $\tilde{N}_\infty(\omega, z)$.

3 Random symbol

We will need a notion of random symbols to express the asymptotic expansion formula.

Given an r -dimensional Wiener space (\mathbb{W}, \mathbb{P}) over time interval $[0, 1]$, the Cameron-Martin subspace is denoted by H . We will assume that the probability space (Ω, \mathcal{F}, P) admits the structure such that $\Omega = \Omega' \times \mathbb{W}$, $\mathcal{F} = \mathcal{F}' \otimes \mathbf{B}(\mathbb{W})$ and $P = P' \times \mathbb{P}$ for some probability space $(\Omega', \mathcal{F}', P')$, and consider the partial Malliavin calculus based on the shifts in H . See Ikeda and Watanabe [9], Nualart [15] for the Malliavin calculus. $\mathbb{D}_{\ell, p}$ denotes the Sobolev space on Ω with indices $\ell \in \mathbb{R}$ and $p \in (1, \infty)$. Let $\mathbb{D}_{\ell, \infty} = \cap_{p \geq 2} \mathbb{D}_{\ell, p}$.

Let $\ell, \mathbf{m} \in \mathbb{Z}_+$. Denote by $\mathcal{C}(\ell, \mathbf{m})$ the set of functions $c : \mathbb{R}^d \rightarrow \mathbb{D}_{\ell, \infty}$ satisfying the following conditions:

²[B1](i)=[R1](ii), [B1](ii)=[R1](iii) in [31].

(i) $c \in C^{\mathbf{m}}(\mathbb{R}^d; \mathbb{D}_{\ell, \infty})$, that is, for every $p \geq 2$, $c : \mathbb{R}^d \rightarrow \mathbb{D}_{\ell, p}$ is Fréchet differentiable \mathbf{m} times in z and all the derivatives (each element taking values in $\mathbb{D}_{\ell, \infty}$) up to order \mathbf{m} are continuous.

(ii) For every $p \geq 2$,

$$\|\partial_z^\mu c(z)\|_{\ell, p} \leq C_{\ell, p}(1 + |z|)^{C_{\ell, p}} \quad (z \in \mathbb{R}^d, \mu \in \mathbb{Z}_+^d \text{ with } |\mu| \leq \mathbf{m})$$

for some constant $C_{\ell, p}$ depending on ℓ and p .

Note that if $c = c(z)$ does not depend on z , then this condition is reduced to that $\|c\|_{\ell, p} < \infty$. The sum of the elements of $\alpha \in \mathbb{Z}_+^d$, i.e., the length of α , is denoted by $|\alpha|$.

Let $\ell, \mathbf{m}, \mathbf{n} \in \mathbb{Z}_+$. A function $\varsigma : \Omega \times \mathbb{R}^d \times (i\mathbb{R}^d) \times (i\mathbb{R}^{d_1}) \rightarrow \mathbb{C}$ is called a *random symbol*. We say that a random symbol ς is of *class* $(\ell, \mathbf{m}, \mathbf{n})$ if ς admits a representation

$$\varsigma(z, iu, iv) = \sum_j c_j(z)(iu)^{m_j}(iv)^{n_j} \quad (\text{finite sum}) \quad (4)$$

for some $c_j \in \mathcal{C}(\ell, |m_j|)$, $m_j \in \mathbb{Z}_+^d$ ($|m_j| \leq \mathbf{m}$) and $n_j \in \mathbb{Z}_+^{d_1}$ ($|n_j| \leq \mathbf{n}$). We denote by $\mathcal{S}(\ell, \mathbf{m}, \mathbf{n})$ the set of random symbols of class $(\ell, \mathbf{m}, \mathbf{n})$.

The Malliavin covariance matrix of a multi-dimensional functional F is denoted by σ_F , and we write $\Delta_F = \det \sigma_F$. Suppose that $\ell \geq \ell_0 := 2([d_1/2] + 1 + [(n+1)/2])^3$ and the following conditions are satisfied:

(i) $F_\infty \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^{d_1})$, $W_\infty \in \mathbb{D}_{\ell, \infty}(\mathbb{R}^d)$ and $C_\infty \in \mathbb{D}_{\ell, \infty}(\mathbb{R}^d \otimes \mathbb{R}^d)$;

(ii) $\Delta_{F_\infty}^{-1}$, $\det C_\infty^{-1} \in \cap_{p \geq 2} L^p$;

(iii) $\varsigma \in \mathcal{S}(\ell, \mathbf{m}, \mathbf{n})$.

For the random symbol $\varsigma \in \mathcal{S}(\ell, \mathbf{m}, \mathbf{n})$ taking the form of (4), we define the adjoint operator ς^* of ς as follows. It applies to $\phi(z; W_\infty, C_\infty)\delta_x(F_\infty)$ as

$$\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty)\delta_x(F_\infty) \right\} = \sum_j (-\partial_z)^{m_j} (-\partial_x)^{n_j} \left(c_j(z)\phi(z; W_\infty, C_\infty)\delta_x(F_\infty) \right). \quad (5)$$

Here $\phi(z; \mu, C)$ is the normal density with mean vector μ and covariance matrix C , and the derivatives are interpreted as the Fréchet derivatives in the space of the generalized Wiener functionals of Watanabe ([26], [9]). By assumptions, $(-\partial_z)^{m_j}\phi(z; W_\infty, C_\infty) \in \mathbb{D}_{\ell, \infty}$ and $(-\partial_x)^{n_j}\delta_x(F_\infty) = (\partial^{n_j}\delta_x)(F_\infty) \in \mathbb{D}_{-\ell, \infty}$. p^{F_∞} denotes the density of F_∞ . The generalized expectation of the formula (5) gives a formula with the usual expectation:

$$\begin{aligned} & E \left[\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty)\delta_x(F_\infty) \right\} \right] \\ &= \sum_j (-\partial_z)^{m_j} (-\partial_x)^{n_j} E \left[c_j(z)\phi(z; W_\infty, C_\infty)\delta_x(F_\infty) \right] \\ &= \sum_j (-\partial_z)^{m_j} (-\partial_x)^{n_j} \left(E [c_j(z)\phi(z; W_\infty, C_\infty) | F_\infty = x] p^{F_\infty}(x) \right) \\ &=: E \left[\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \Big|_{F_\infty = x} p^{F_\infty}(x) \right\} \right]. \end{aligned}$$

For $m' \in \mathbb{Z}_+^d$ and $n' \in \mathbb{Z}_+^{d_1}$,

$$\begin{aligned} & \sup_{z, x} \left| z^{m'} x^{n'} E \left[(-\partial_z)^\mu c_j(z) \cdot (-\partial_z)^{m_j - \mu} \phi(z; W_\infty, C_\infty) (-\partial_x)^{n_j} \delta_x(F_\infty) \right] \right| \\ & \leq \sup_{z, x} \| (-\partial_z)^\mu c_j(z) \|_{\ell, p} \| z^{m'} (-\partial_z)^{m_j - \mu} \phi(z; W_\infty, C_\infty) \|_{\ell, p_1} \| F_\infty^{n'} \|_{\ell, p_2} \| (-\partial_x)^{n_j} \delta_x(F_\infty) \|_{-\ell, p_3} \\ & \lesssim \sup_{z, x} \{ (1 + |z|)^{C_{\ell, p}} \cdot (1 + |z|)^{-C_{\ell, p}} \cdot 1 \} \\ & < \infty, \end{aligned} \quad (6)$$

³The number $\ell_0 = 2[(n + d_1 + 2)/2]$ is possible to use for this ℓ_0 if we regard $\partial^{n_j}\delta_x$ as a Schwartz distribution.

where $p, p_1, p_2, p_3 > 1$ with $p^{-1} + p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$. Consequently, we can apply the Fourier transform to obtain

$$\begin{aligned} & \mathcal{F}_{(z,x)} \left[\left[E \left[\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \delta_x(F_\infty) \right\} \right] \right] \right] (u, v) \\ &= \int \exp(iu \cdot z + iv \cdot x) E \left[\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \delta_x(F_\infty) \right\} \right] dz dx \\ &= E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iF_\infty[v]) \phi(z; W_\infty, C_\infty) \varsigma(z, iu, iv) dz \right]. \end{aligned} \quad (7)$$

More generally, duality argument also yields

$$\mathcal{F}_{(z,x)} \left[\left[\varsigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \delta_x(F_\infty) \right\} \right] \right] (u, v) = \mathcal{F}_z \left[\left[\varsigma(z, iu, iv) \phi(z; W_\infty, C_\infty) e^{iu \cdot F_\infty} \right] \right] (u).$$

4 Asymptotic expansion formula

Nondegeneracy of the targeted distribution is indispensable for asymptotic expansion. However, the complete nondegeneracy, that implies absolute continuity, is not necessary, nor can we assume in statistical inference. For example, the maximum likelihood estimator does not admit a density in general; even existence of itself is not ensured on the whole probability space. Besides, the complete nondegeneracy is often hard to prove although it would be possible, and this restricts applications of the result. On the other hand, partial nondegeneracy is easier to work with. There the localization method plays an essential role. The localization is realized through a sequence of \mathcal{F} -measurable truncation functionals ξ_n . In applications, we construct a suitable ξ_n to validate necessary nondegeneracy of the functional in question.

We will assume the following conditions.⁴

[B2]_ℓ (i) $F_\infty \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^{d_1})$, $W_\infty \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$ and $C_\infty \in \mathbb{D}_{\ell,\infty}(\mathbb{R}^d \otimes \mathbb{R}^d)$.

(ii) $M_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$, $F_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^{d_1})$, $W_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$, $C_n \in \mathbb{D}_{\ell,\infty}(\mathbb{R}^d \otimes \mathbb{R}^d)$, $N_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$ and $\xi_n \in \mathbb{D}_{\ell,\infty}(\mathbb{R})$. Moreover,

$$\sup_{n \in \mathbb{N}} \left\{ \|M_n\|_{\ell+1,p} + \|\mathring{C}_n\|_{\ell,p} + \|\mathring{W}_n\|_{\ell+1,p} + \|\mathring{F}_n\|_{\ell+1,p} + \|N_n\|_{\ell+1,p} + \|\xi_n\|_{\ell,p} \right\} < \infty$$

for every $p \geq 2$.

[B3] (i) $\lim_{n \rightarrow \infty} P[|\xi_n| \leq \frac{1}{2}] = 1$.

(ii) $|C_n - C_\infty| > r_n^{1-a}$ implies $|\xi_n| \geq 1$, where $a \in (0, 1/3)$ is a constant.

(iii) For every $p \geq 2$,

$$\limsup_{n \rightarrow \infty} E \left[1_{\{|\xi_n| \leq 1\}} \Delta_{(M_n + W_\infty, F_\infty)}^{-p} \right] < \infty$$

and moreover $\det C_\infty^{-1} \in \cap_{p \geq 2} L^p$.

We have $\Delta_{F_\infty}^{-1} \in \cap_{p \geq 2} L^p$ by Fisher's inequality and Fatou's lemma.

We will give an asymptotic expansion formula to approximate the joint distribution of (Z_n, F_n) . Let

$$\begin{aligned} \underline{\sigma}(z, iu, iv) &= \frac{1}{2} \tilde{C}_\infty(z)^{j,k} (iu_j)(iu_k) + \tilde{W}_\infty(z)^j (iu_j) \\ &\quad + \tilde{N}_\infty(z)^j (iu_j) + \tilde{F}_\infty(z)^l (iv_l) \end{aligned} \quad (8)$$

⁴[B2]_ℓ(i),(ii) are [S]_{ℓ,m,n}(i),(ii) of [31], respectively. [B3](i),(ii),(iii) are [S]_{ℓ,m,n}(iii),(iv),(v) of [31], respectively.

for $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^{d_1}$.⁵

The symbol (8) is the adaptive random random symbol, which corresponds to the second-order correction term of the asymptotic expansion in the normal limit ([28]). In the mixed normal limit case, we need an additional one referred to as the anticipative random symbol and denoted by $\bar{\sigma}(z, iu, iv)$. As will be seen in applications in this paper, the anticipative random symbol is given by the Malliavin derivatives, which shows non-classical nature of the present asymptotic expansion beyond the standard invariance principle.

We shall define the anticipative random symbol. Let

$$\Psi_\infty(u, v) = \exp \left\{ iW_\infty[u] - \frac{1}{2}C_\infty[u^{\otimes 2}] + iF_\infty[v] \right\}.$$

Moreover, set $L_t^n(u) = e_t^n(u) - 1$ with

$$e_t^n(u) = \exp \left(iM_t^n[u] + \frac{1}{2}C_t^n[u^{\otimes 2}] \right).$$

We write $\mathfrak{D}^\alpha = i^{-|\alpha|} \partial^\alpha$, $\mathfrak{D}_u^{\alpha_1} = i^{-|\alpha_1|} \partial_u^{\alpha_1}$, $\mathfrak{D}_v^{\alpha_2} = i^{-|\alpha_2|} \partial_v^{\alpha_2}$ and $\mathfrak{D}^\alpha = \mathfrak{D}_u^{\alpha_1} \mathfrak{D}_v^{\alpha_2}$ for the multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^{\check{d}}$, $\check{d} = d + d_1$. Let $\psi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\psi(x) = 1$ if $|x| \leq 1/2$ and $\psi(x) = 0$ if $|x| \geq 1$ and let $\psi_n = \psi(\xi_n)$ for $n \in \mathbb{N}$. Furthermore, let

$$\Phi_n^{2,\alpha}(u, v) = \mathfrak{D}^\alpha E[L_1^n(u) \Psi_\infty(u, v) \psi_n].$$

If ψ_n is equal to one, in fact it is usually so without large deviation probability, and W_∞ , C_∞ and F_∞ are constants, as this is the case in the normal limit case, then $\Phi_n^{2,\alpha} = 0$ since $L^n(u)$ becomes a mean zero martingale. However $\Phi_n^{2,\alpha}$ does not vanish in general. We may intuitively say that $\Phi_n^{2,\alpha}$ measures the torsion of martingales under the shift of measure P by $\Psi_\infty(u, v)$. The effect of this torsion appears quite differently, depending on the cases. Thus, we will treat the effect in a slightly abstract shape for a while. The adaptive random symbol describes it as (ii) of the following condition.⁶ Let $\ell_* = 2[d_1/2] + 4$.

[B4] _{$\ell, \mathbf{m}, \mathbf{n}$} (i) $\underline{\sigma} \in \mathcal{S}(\ell_*, 2, 1)$.⁷

(ii) There exists a random symbol $\bar{\sigma} \in \mathcal{S}(\ell, \mathbf{m}, \mathbf{n})$ admitting a representation

$$\bar{\sigma}(iu, iv) = \sum_j c_j (iu)^{m_j} (iv)^{n_j} \quad (\text{finite sum})$$

for some random variables c_j and satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} r_n^{-1} \Phi_n^{2,\alpha}(u, v) &= \mathfrak{D}^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iF_\infty[v]) \phi(z; W_\infty, C_\infty) dz \bar{\sigma}(iu, iv) \right] \\ &= \mathfrak{D}^\alpha E \left[\exp \left(iW_\infty[u] - \frac{1}{2}C_\infty[u^{\otimes 2}] + iF_\infty[v] \right) \bar{\sigma}(iu, iv) \right] \end{aligned} \quad (9)$$

for $u \in \mathbb{R}^d$, $v \in \mathbb{R}^{d_1}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^{\check{d}}$.

Set $\tilde{\Phi}^{2,\alpha}(u, v) = \lim_{n \rightarrow \infty} r_n^{-1} \Phi_n^{2,\alpha}(u, v)$.

Remark 1. It is also possible to consider a more general random symbol $\bar{\sigma}(z, x, iu, iv)$.

Remark 2. As mentioned above, $\tilde{\Phi}^{2,\alpha}$ vanishes in the classical case of deterministic $\Psi_\infty(u, v)$ thanks to the local martingale property of $L_t^n(u)$. Non-vanishing case will appear later.

⁵Einstein's rule for repeated indices.

⁶[B4] _{$\ell, \mathbf{m}, \mathbf{n}$} (i),(ii) are [S] _{$\ell, \mathbf{m}, \mathbf{n}$} (vi),(vii) of [31].

⁷The number $\ell_* = 2[(d_1 + 3)/2]$ is possible as the previous footnote.

The full random symbol for the second-order terms is

$$\sigma := \underline{\sigma} + \bar{\sigma}. \quad (10)$$

In order to approximate the joint local density of (Z_n, F_n) , we use the density function

$$\begin{aligned} p_n(z, x) &= E \left[\phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right] p^{F_\infty}(x) \\ &\quad + r_n E \left[\sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right\} p^{F_\infty}(x) \right]. \end{aligned}$$

Note that $p_n(z, x)$ is well defined under [B2] $_\ell$, [B3] and [B4] $_{\ell, \mathbf{m}, \mathbf{n}}$ when $\ell \geq \ell_0$. We should remark that $p_n(z, x)$ is written in terms of the conditional expectation given F_∞ . This suggests that conditioning by F_∞ or equivalently by F_n under truncation is essentially used in validation of the formula. With Watanabe's delta functional, we can write

$$\begin{aligned} p_n(z, x) &= E \left[\phi(z; W_\infty, C_\infty) \delta_x(F_\infty) \right] \\ &\quad + r_n E \left[\sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \delta_x(F_\infty) \right\} \right]. \end{aligned} \quad (11)$$

For $M, \gamma > 0$, let $\mathcal{E}(M, \gamma)$ denote the set of measurable functions $f : \mathbb{R}^{\check{d}} \rightarrow \mathbb{R}$ satisfying $|f(z, x)| \leq M(1 + |z| + |x|)^\gamma$. Let

$$\Delta_n(f) = \left| E[f(Z_n, F_n)] - \int f(z, x) p_n(z, x) dz dx \right|.$$

Let $\Lambda_n^0(d, q) = \{u \in \mathbb{R}^d; |u| \leq r_n^{-q}\}$, where $q = (1 - a)/2 \in (1/3, 1/2)$. Let

$$\epsilon(k, n) = \max_{\alpha: |\alpha| \leq k} \frac{1}{(2\pi)^{\check{d}}} r_n \int_{\Lambda_n^0(\check{d}, q)} |r_n^{-1} \Phi_n^{2, \alpha}(u, v) - \tilde{\Phi}^{2, \alpha}(u, v)| du dv.$$

The following theorem gives an extension of Theorem 4 of [28].

Theorem 1. *Let $\ell = \ell_0 \vee (\check{d} + 3)$. Suppose that [B1], [B2] $_\ell$, [B3] and [B4] $_{\ell, \mathbf{m}, \mathbf{n}}$ are fulfilled. Let $M, \gamma \in (0, \infty)$ and $\theta \in (0, 1)$ be arbitrary numbers. Then*

(a) *there exist constants $C_1 = C(M, \gamma, \theta)$ and $C_2 = C(M, \gamma)$ such that*

$$\sup_{f \in \mathcal{E}(M, \gamma)} \Delta_n(f) \leq C_1 P \left[|\xi_n| > \frac{1}{2} \right]^\theta + C_2 \epsilon([\gamma + \check{d}] + 1, n) + o(r_n)$$

as $n \rightarrow \infty$.

(b) *If*

$$\sup_n \sup_{(u, v) \in \Lambda_n^0(\check{d}, q)} r_n^{-1} |(u, v)|^{\check{d}+1-\epsilon} |\Phi_n^{2, \alpha}(u, v)| < \infty \quad (12)$$

for every $\alpha \in \mathbb{Z}_+^{\check{d}}$ and some $\epsilon = \epsilon(\alpha) \in (0, 1)$, then for some constant $C_1 = C(M, \gamma, \theta)$,

$$\sup_{f \in \mathcal{E}(M, \gamma)} \Delta_n(f) \leq C_1 P \left[|\xi_n| > \frac{1}{2} \right]^\theta + o(r_n) \quad (13)$$

as $n \rightarrow \infty$.

In Theorem 1, \mathbf{m} is some number, which puts restriction when (12) is verified. Next, we shall present a version of Theorem 1. Let s_n be a positive random variable on Ω .

[B2']_ℓ (i) $F_\infty \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^{d_1})$, $W_\infty \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$ and $C_\infty \in \mathbb{D}_{\ell,\infty}(\mathbb{R}^d \otimes \mathbb{R}^d)$.

(ii) $M_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$, $F_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^{d_1})$, $W_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$, $C_n \in \mathbb{D}_{\ell,\infty}(\mathbb{R}^d \otimes \mathbb{R}^d)$, $N_n \in \mathbb{D}_{\ell+1,\infty}(\mathbb{R}^d)$ and $s_n \in \mathbb{D}_{\ell,\infty}(\mathbb{R})$. Moreover,

$$\sup_{n \in \mathbb{N}} \left\{ \|M_n\|_{\ell+1,p} + \|\mathring{C}_n\|_{\ell,p} + \|\mathring{W}_n\|_{\ell+1,p} + \|\mathring{F}_n\|_{\ell+1,p} + \|N_n\|_{\ell+1,p} + \|s_n\|_{\ell,p} \right\} < \infty$$

for every $p \geq 2$.

[B3'] (i) $P[\Delta_{(M_n+W_\infty, F_\infty)} < s_n] = O(r_n^{1+\kappa})$ as $n \rightarrow \infty$ for some $\kappa > 0$.

(ii) For every $p \geq 2$,

$$\limsup_{n \rightarrow \infty} E[s_n^{-p}] < \infty$$

and moreover $\det C_\infty^{-1} \in \cap_{p \geq 2} L^p$.

Theorem 2. Let $\ell = \ell_0 \vee (\check{d} + 3)$. Suppose that [B1], [B2']_ℓ, [B3'] and [B4]_{ℓ,m,n} are fulfilled. Then for any $M, \gamma \in (0, \infty)$,

(a) there exists a constant C such that

$$\sup_{f \in \mathcal{E}(M, \gamma)} \Delta_n(f) \leq C\epsilon([\gamma + \check{d}] + 1, n) + o(r_n)$$

as $n \rightarrow \infty$.

(b) If (12) is satisfied, then

$$\sup_{f \in \mathcal{E}(M, \gamma)} \Delta_n(f) = o(r_n)$$

as $n \rightarrow \infty$.

The above results can apply to the expansion of the distribution of functions of (Z_n, F_n) , in particular, Z_n/F_n and $Z_n/\sqrt{F_n}$.

5 Proof of Theorems 1 and 2

5.1 Decomposition of the joint characteristic function

Let $\check{Z}_n = (Z_n, F_n)$. Then according to the notation of the multi-index, $\check{Z}_n^\alpha = Z_n^{\alpha_1} F_n^{\alpha_2}$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^d \times \mathbb{Z}_+^{d_1} = \mathbb{Z}_+^{\check{d}}$, $\check{d} = d + d_1$. Let

$$\hat{g}_n^\alpha(u, v) = E[\psi_n \check{Z}_n^\alpha \exp(iZ_n[u] + iF_n[v])]$$

for $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^{d_1}$. We denote

$$g_n^\alpha(z, x) = \frac{1}{(2\pi)^{\check{d}}} \int_{\mathbb{R}^{\check{d}}} e^{-iu \cdot z - iv \cdot x} \hat{g}_n^\alpha(u, v) du dv$$

if the integral on the right-hand side exists.

Let

$$\mathring{e}(x) = \int_0^1 e^{sx} ds,$$

then $e^x = 1 + x \overset{\circ}{e}(x)$. For

$$\Psi_n(u, v) = \exp \left\{ iW_n[u] - \frac{1}{2}C_n[u^{\otimes 2}] + iF_n[v] \right\},$$

we have

$$\begin{aligned} \epsilon_n(u, v) &= \log(\Psi_n(u, v)\Psi_\infty(u, v)^{-1}) + ir_n N_n[u] \\ &= -\frac{1}{2}(C_n - C_\infty)[u^{\otimes 2}] + i(F_n[v] - F_\infty[v]) \\ &\quad + i(W_n[u] - W_\infty[u]) + ir_n N_n[u]. \end{aligned}$$

Then we have

$$\begin{aligned} \hat{g}_n^\alpha(u, v) &= \mathfrak{D}^\alpha E[\Psi_\infty(u, v)\psi_n] + \mathfrak{D}^\alpha E \left[e_1^n(u)\Psi_\infty(u, v)\epsilon_n(u, v) \overset{\circ}{e}(\epsilon_n(u, v))\psi_n \right] \\ &\quad + \mathfrak{D}^\alpha E[L_1^n(u)\Psi_\infty(u, v)\psi_n] \\ &=: \Phi_n^{0, \alpha}(u, v) + \Phi_n^{1, \alpha}(u, v) + \Phi_n^{2, \alpha}(u, v). \end{aligned} \tag{14}$$

Set

$$B_t^n(u, v) = e_t^n(u)\Psi_\infty(u, v).$$

Then

$$\begin{aligned} B_t^n(u, v) &= \exp \left(iM_t^n[u] + \frac{1}{2}C_t^n[u^{\otimes 2}] \right) \exp \left\{ iW_\infty[u] - \frac{1}{2}C_\infty[u^{\otimes 2}] + iF_\infty[v] \right\} \\ &= \exp(iM_t^n[u]) \exp(iW_\infty[u] + iF_\infty[v]) \exp \left(\frac{1}{2}(C_t^n - C_\infty)[u^{\otimes 2}] \right). \end{aligned}$$

Condition [B3] (ii) implies

$$\begin{aligned} |B_t^n(u, v)| &= \exp \left(\frac{1}{2}(C_t^n - C_1^n)[u^{\otimes 2}] \right) \exp \left(\frac{1}{2}(C_n - C_\infty)[u^{\otimes 2}] \right) \\ &\leq \exp \left(\frac{1}{2}r_n^{1-a}|u|^2 \right) \end{aligned} \tag{15}$$

and

$$\Re \epsilon_n(u, v) \leq \frac{1}{2}r_n^{1-a}|u|^2 \tag{16}$$

whenever $\psi_n > 0$.

We shall consider the second-order term of type I.

Lemma 1. *Suppose that [B1] and [B2] $_\ell$ for some $\ell \geq 0$ and [B3] (i), (ii)⁸ are fulfilled. Then the limit*

$$\tilde{\Phi}^{1, \alpha}(u, v) = \lim_{n \rightarrow \infty} r_n^{-1} \Phi_n^{1, \alpha}(u, v)$$

exists and takes the form

$$\begin{aligned} \tilde{\Phi}^{1, \alpha}(u, v) &= \mathfrak{D}^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iF_\infty[v]) \right. \\ &\quad \cdot \left(-\frac{1}{2}\tilde{C}_\infty(z)[u^{\otimes 2}] + i\tilde{W}_\infty(z)[u] + i\tilde{N}_\infty(z)[u] + i\tilde{F}_\infty(z)[v] \right) \phi(z; W_\infty, C_\infty) dz \Big] \end{aligned}$$

for every $\alpha \in \mathbb{Z}_+^d$.

⁸It suffices to assume [R1] and [R2] of [31].

Proof. Since the family of functions $u \mapsto B_1^n(u, v)\psi_n$ ($n \in \mathbb{N}$) is uniformly (in n) locally (in u) bounded due to (15), we have, by using the weak convergence condition and (16),

$$\begin{aligned}
\tilde{\Phi}^{1,\alpha}(u, v) &= \lim_{n \rightarrow \infty} r_n^{-1} \Phi_n^{1,\alpha}(u, v) \\
&= \lim_{n \rightarrow \infty} r_n^{-1} \vartheta^\alpha E \left[\exp(iM_1^n[u]) \exp(iW_\infty[u] + iF_\infty[v]) \epsilon_n(u, v) \exp\left(\frac{1}{2}(C_n - C_\infty)[u^{\otimes 2}]\right) \overset{\circ}{e}(\epsilon_n(u, v)) \psi_n \right] \\
&= \lim_{n \rightarrow \infty} E \left[\vartheta^\alpha \left\{ \exp(iM_1^n[u]) \exp(iW_\infty[u] + iF_\infty[v]) r_n^{-1} \epsilon_n(u, v) \right\} \exp\left(\frac{1}{2}(C_n - C_\infty)[u^{\otimes 2}]\right) \overset{\circ}{e}(\epsilon_n(u, v)) \psi_n \right] \\
&= \vartheta^\alpha E \left[\exp(iM_1^\infty[u] + iW_\infty[u] + iF_\infty[v]) \right. \\
&\quad \cdot \left. \left(-\frac{1}{2} \overset{\circ}{C}_\infty[u^{\otimes 2}] + i \overset{\circ}{W}_\infty[u] + iN_\infty[u] + i \overset{\circ}{F}_\infty[v] \right) \right].
\end{aligned}$$

The last expression is equal to

$$\begin{aligned}
&\vartheta^\alpha E \left[\exp(iM_1^\infty[u] + iW_\infty[u] + iF_\infty[v]) \right. \\
&\quad \cdot \left. \left(-\frac{1}{2} \mathbb{E}_{\tilde{\mathcal{F}}}[\overset{\circ}{C}_\infty][u^{\otimes 2}] + i \mathbb{E}_{\tilde{\mathcal{F}}}[\overset{\circ}{W}_\infty][u] + i \mathbb{E}_{\tilde{\mathcal{F}}}[N_\infty][u] + i \mathbb{E}_{\tilde{\mathcal{F}}}[\overset{\circ}{F}_\infty][v] \right) \right] \\
&= \vartheta^\alpha E \left[\exp(iM_\infty[u] + iW_\infty[u] + iF_\infty[v]) \right. \\
&\quad \cdot \left. \left(-\frac{1}{2} \check{C}_\infty(M_\infty)[u^{\otimes 2}] + i \check{W}_\infty(M_\infty)[u] + i \check{N}_\infty(M_\infty)[u] + i \check{F}_\infty(M_\infty)[v] \right) \right] \\
&= \vartheta^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iW_\infty[u] + iF_\infty[v]) \right. \\
&\quad \cdot \left. \left(-\frac{1}{2} \check{C}_\infty(z)[u^{\otimes 2}] + i \check{W}_\infty(z)[u] + i \check{N}_\infty(z)[u] + i \check{F}_\infty(z)[v] \right) \phi(z; 0, C_\infty) dz \right] \\
&= \vartheta^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iF_\infty[v]) \right. \\
&\quad \cdot \left. \left(-\frac{1}{2} \check{C}_\infty(z)[u^{\otimes 2}] + i \check{W}_\infty(z)[u] + i \check{N}_\infty(z)[u] + i \check{F}_\infty(z)[v] \right) \phi(z; W_\infty, C_\infty) dz \right]
\end{aligned}$$

by \mathcal{F} -conditional expectation, which is what we desired. \square

In particular,

$$\tilde{\Phi}^{1,\alpha}(u, v) = \vartheta^\alpha \tilde{\Phi}^{1,0}(u, v). \quad (17)$$

Under $[\text{B4}]_{\ell, m, n}$ (i), the conditions just before (5) are satisfied for $\varsigma = \underline{\sigma}$, and

$$\begin{aligned}
\mathcal{F}_{(u,v)}^{-1}[\tilde{\Phi}^{1,0}](z, x) &= \frac{1}{2} \partial_{z_j} \partial_{z_k} \left(E[\check{C}_\infty(z)^{j,k} \phi(z; W_\infty, C_\infty) | F_\infty = x] \right) p^{F_\infty}(x) \\
&\quad - \partial_{z_j} \left(E[\check{W}_\infty(z)^j \phi(z; W_\infty, C_\infty) | F_\infty = x] \right) p^{F_\infty}(x) \\
&\quad - \partial_{z_j} \left(E[\check{N}_\infty(z)^j \phi(z; W_\infty, C_\infty) | F_\infty = x] \right) p^{F_\infty}(x) \\
&\quad - \partial_{x_l} \left(E[\check{F}_\infty(z)^l \phi(z; W_\infty, C_\infty) | F_\infty = x] p^{F_\infty}(x) \right) \\
&= E \left[\underline{\sigma}(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right\} p^{F_\infty}(x) \right].
\end{aligned}$$

We can say that the random symbol $\underline{\sigma}$ corresponds to the case where the martingale central limit theorem occurs without conditioning. On the other hand, we need another random symbol which reflects the deviation of the martingale in question from a *real martingale* under conditioning. The Fourier inversion $\mathcal{F}_{(u,v)}^{-1}[\tilde{\Phi}^{2,0}]$ has a random symbol $\bar{\sigma}(\partial_z, \partial_x)$ in that

$$\mathcal{F}_{(u,v)}^{-1}[\tilde{\Phi}^{2,0}](z, x) = E \left[\bar{\sigma}(\partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right\} p^{F_\infty}(x) \right].$$

See (7) and (9) for this equality.

5.2 Estimates of error bounds

We shall investigate the approximation error of $p_n(z, x)$ to the joint local density of (Z_n, F_n) . The error bounds depend on the smoothness of the distribution $\mathcal{L}\{(Z_n, F_n)\}$, and the arguments will require a tool to evaluate it quantitatively. The conditions are written in terms of the Malliavin calculus since it is convenient in practical uses while more primitive expression of them would be possible without it if we would admit more cumbersome descriptions.

Let

$$\begin{aligned} h_n^0(z, x) &= E \left[\psi_n \phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right] p^{F_\infty}(x) \\ &\quad + r_n E \left[\sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W_\infty, C_\infty) \middle| F_\infty = x \right\} p^{F_\infty}(x) \right]. \end{aligned}$$

and let

$$h_n^\alpha(z, x) = (z, x)^\alpha h_n^0(z, x).$$

Lemma 2. *Let $\ell \geq 2$. Suppose that $[B2]_\ell$ and $[B3]$ are fulfilled. Then*

$$\sup_n \sup_{u \in \Lambda_n^0(d, q)} \sup_{v \in \Lambda_n^0(d_1, 2q)} r_n^{-1} |(u, v)|^{\ell-2} |\Phi_n^{1, \alpha}(u, v)| < \infty.$$

Moreover,

$$\sup_{u \in \mathbb{R}^d} \sup_{v \in \mathbb{R}^{d_1}} |(u, v)|^{\ell-2} |\tilde{\Phi}^{1, \alpha}(u, v)| < \infty.$$

Proof. By definition,

$$\begin{aligned} r_n^{-1} \Phi_n^{1, \alpha}(u, v) &= \partial^\alpha E \left[\exp \left(i(M_1^n + W_\infty)[u] + iF_\infty[v] \right) \right. \\ &\quad \cdot r_n^{-1} \epsilon_n(u, v) \exp \left(\frac{1}{2} (C_n - C_\infty)[u^{\otimes 2}] \right) \left. \right] \overset{\circ}{e}(\epsilon_n(u, v)) \psi_n \\ &= E \left[\exp \left(i(M_1^n + W_\infty)[u] + iF_\infty[v] \right) \right. \\ &\quad \cdot p_{2,1,\alpha} \left(u, v; r_n, M_1^n, W_\infty, F_\infty, N_n, \overset{\circ}{C}_n, \overset{\circ}{W}_n, \overset{\circ}{F}_n \right) \left. \right] \psi_n, \end{aligned} \tag{18}$$

where $p_{2,1,\alpha}$ is a smooth function such that

$$|p_{2,1,\alpha}(u, v; x)| \leq C_\alpha (1 + |u|^2 + |v|)(1 + |x|)^{C_\alpha} \quad (u \in \mathbb{R}^d, v \in \mathbb{R}^{d_1}, x \in \mathbb{R}^{d^2+4d+2d_1+1})$$

for some constant C_α , and moreover the derivative of $p_{2,1,\alpha}$ of any order admits the same type estimate. It should be noted that in the expectation on the right-hand side of (18), we do not need an exponential like factor in $p_{2,1,\alpha}$, due to the truncation by ψ_n .

We will apply the integration-by-parts formula ℓ -times for the pull-back of the function $f(z, x) = e^{iu \cdot z + iv \cdot x}$ by taking the advantage of the uniform nondegeneracy of the functional $(M_n + W_\infty, F_\infty)$ under ψ_n . Then, by truncation by ψ_n and the restriction of the region of (u, v) , the functional in the expectation of (18) is essentially quadratic in u and linear in v . It should be noted that all variables related to u or v are differentiated at the same time when applying the IBP-formula, therefore the index of differentiability should be common.

The second inequality follows from the first one if one takes the limit in n . \square

Define the $\check{d} \times \check{d}$ random matrix R'_n by

$$R'_n = \sigma_{Q_n}^{-1} (r_n \langle DQ_n, DR_n \rangle + r_n \langle DR_n, DQ_n \rangle + r_n^2 \langle DR_n, DR_n \rangle),$$

where $Q_n = (M_n + W_\infty, F_\infty)$ and $R_n = (\overset{\circ}{W}_n + N_n, \overset{\circ}{F}_n)$. Obviously

$$\sigma_{(Z_n, F_n)} = \sigma_{Q_n} (I_{\check{d}} + R'_n). \quad (19)$$

Let $\xi'_n = r_n^{-1} |R'_n|^2$. We redefine ψ_n by

$$\psi_n = \psi(\xi_n) \psi(\xi'_n). \quad (20)$$

$\Phi_n^{j,\alpha}$ ($j = 0, 1, 2$), g_n^α and h_n^α will be defined for ψ_n given in (20). The following is an extension of the expansion of the local density in the normal limit case; see Theorems 3 and 5 of [28].

Lemma 3. *Let $\ell = \ell_0 \vee (\check{d} + 3)$. Suppose that [B1], [B2] $_\ell$, [B3] and [B4] $_{\ell, \mathbf{m}, \mathbf{n}}$ are satisfied. Then for every $k \in \mathbb{Z}_+$,*

$$\sup_{(z,x) \in \mathbb{R}^{\check{d}+d_1}} |(z,x)^k (g_n^0(z,x) - h_n^0(z,x))| \leq \epsilon(k,n) + o(r_n). \quad (21)$$

Furthermore, if (12) is satisfied for every $\alpha \in \mathbb{Z}_+^{\check{d}}$ and some $\epsilon = \epsilon(\alpha) \in (0, 1)$, then $\epsilon(k,n) = o(r_n)$ for every $k \in \mathbb{Z}_+$.

Proof. By definition,

$$\hat{h}_n^\alpha(u,v) = \mathfrak{D}^\alpha E[\Psi_\infty(u,v) \psi_n] + r_n \mathfrak{D}^\alpha \tilde{\Phi}^{1,0}(u,v) + r_n \mathfrak{D}^\alpha \tilde{\Phi}^{2,0}(u,v).$$

Moreover, we have (14) for $\hat{g}_n^\alpha(u,v)$. Applying the IBP formula ℓ -times under truncation by ψ_n to the definition of \hat{g}_n^α , we see that for every $\alpha \in \mathbb{Z}_+^{\check{d}}$, there exist $C > 0$ and $n^* \in \mathbb{N}$ such that

$$|\hat{g}_n^\alpha(u,v)| \leq C(1 + |u| + |v|)^{-\ell} \quad (22)$$

for all $(u,v) \in \mathbb{R}^{\check{d}}$ and $n \geq n^*$. In what follows, we will consider only sufficiently large n . Therefore the local densities $g_n^\alpha(z,x)$ are well defined. On the other hand, one can verify the integrability of $\tilde{\Phi}^{2,\alpha}(u,v)$ by using the nondegeneracy of C_∞ , the nondegeneracy of the Malliavin covariance of F_∞ in [B3] and the representation (9). [The aftereffect by the integration-by-parts for F_∞ is absorbed by the nondegeneracy of C_∞ .] The integrability of $\Phi_n^{1,\alpha}(u,v)$ and $\tilde{\Phi}^{1,\alpha}(u,v)$ has been obtained in Lemma 2. The decomposition (14) implies that $\Phi_n^{2,\alpha}$ is integrable for each n . Thus, the decomposition and the estimate of the following inequality based on the Fourier inversion formula are valid:

$$\begin{aligned} \sup_{(z,x) \in \mathbb{R}^{\check{d}}} |g_n^\alpha(z,x) - h_n^\alpha(z,x)| &= \sup_{(z,x) \in \mathbb{R}^{\check{d}}} \frac{1}{(2\pi)^{\check{d}}} \left| \int_{\mathbb{R}^{\check{d}}} e^{-iu \cdot z - iv \cdot x} (\hat{g}_n^\alpha(u,v) - \hat{h}_n^\alpha(u,v)) dudv \right| \\ &\leq \frac{1}{(2\pi)^{\check{d}}} \int_{\mathbb{R}^{\check{d}} \setminus \Lambda_n^0(\check{d},q)} (|\hat{g}_n^\alpha(u,v)| + |\hat{h}_n^\alpha(u,v)|) dudv \\ &\quad + \frac{1}{(2\pi)^{\check{d}}} r_n \int_{\Lambda_n^0(\check{d},q)} |r_n^{-1} \Phi_n^{1,\alpha}(u,v) - \tilde{\Phi}^{1,\alpha}(u,v)| dudv \\ &\quad + \frac{1}{(2\pi)^{\check{d}}} r_n \int_{\Lambda_n^0(\check{d},q)} |r_n^{-1} \Phi_n^{2,\alpha}(u,v) - \tilde{\Phi}^{2,\alpha}(u,v)| dudv. \end{aligned} \quad (23)$$

We remark that $\tilde{\Phi}^{2,\alpha} = \mathfrak{D}^\alpha \tilde{\Phi}^{2,0}$ from (9) as well as $\tilde{\Phi}^{1,\alpha} = \mathfrak{D}^\alpha \tilde{\Phi}^{1,0}$ mentioned at (17).

By (22),

$$\int_{\mathbb{R}^{\check{d}} \setminus \Lambda_n^0(\check{d},q)} |\hat{g}_n^\alpha(u,v)| dudv = O(r_n^{q(\ell-\check{d})}) = O(r_n^{3q}) = o(r_n)$$

since $3q > 1$ due to $a < 1/3$.

As suggested above, we use the Gaussianity in u and the IBP formula in v to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus \Lambda_n^0(\check{d}, q)} |\hat{h}_n^\alpha(u, v)| dudv &\lesssim \int_{\mathbb{R}^d \setminus \Lambda_n^0(\check{d}, q)} |(u, v)|^{-\check{d}-3} dudv \\
&+ r_n \int_{\mathbb{R}^d \setminus \Lambda_n^0(\check{d}, q)} |(u, v)|^{-\check{d}-3+2} dudv \\
&+ r_n \int_{\mathbb{R}^d \setminus \Lambda_n^0(\check{d}, q)} (1 + |u|)^{-k} |v|^{-\{(n+d_1+1)\}+n} dudv \\
&= O(r_n^{3q}) + o(r_n) = o(r_n),
\end{aligned}$$

where $k \geq d+1$ is an arbitrary number. The second term on the right-hand side of (23) is estimated by Lemma 2.

In order to conclude the second assertion of the lemma, to the third term on the right-hand side of (23), obviously, we apply (12) and the inequality which is obtained by its limit as $n \rightarrow \infty$. \square

Proof of Theorem 1. Let $f \in \mathcal{E}(M, \gamma)$. By the estimate (22), we know that the local density g_n^0 exists. In fact g_n^0 is a continuous version of $\{E[\psi_n | \check{Z}_n = (z, x)] dP^{\check{Z}_n}\} / dz dx$, it has moments of any order and

$$E[f(\check{Z}_n) \psi_n] = \int_{\mathbb{R}^d} f(z, x) g_n^0(z, x) dz dx.$$

In particular, the integrability of $f(z, x) g_n^0(z, x)$ is obvious. We have

$$|E[f(\check{Z}_n)] - E[f(\check{Z}_n) \psi_n]| \leq \|f(\check{Z}_n)\|_{L^{p'}} \|1 - \psi_n\|_{L^p}$$

for any dual pair of positive numbers (p, p') with $1/p + 1/p' = 1$. The integrability of $f(z, x) h_n^0(z, x)$ is known from (6) in Section 3 or in particular from Lemma 3, and also we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} f(z, x) g_n^0(z, x) dz dx - \int_{\mathbb{R}^d} f(z, x) h_n^0(z, x) dz dx \right| \\
&\leq \int_{\mathbb{R}^d} |f(z, x)| (1 + |(z, x)|^2)^{-k/2} dz dx \cdot \sup_{(z, x) \in \mathbb{R}^d} |(1 + |(z, x)|^2)^{k/2} (g_n^0(z, x) - h_n^0(z, x))|
\end{aligned}$$

for any $k > \gamma + \check{d}$. Apply a similar estimate as in (6) for $c_j = 1 - \psi_n$ and $p \in (1, 1/\theta)$ to obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} f(z, x) E\left[(1 - \psi_n) \phi(z; W_\infty, C_\infty) \delta_x(F_\infty)\right] dz dx \right| \\
&\leq C \|1 - \psi_n\|_{\ell, p},
\end{aligned}$$

which serves to replace p_n by h_n^0 . Furthermore

$$\|1 - \psi_n\|_{L^p} \leq \|1 - \psi_n\|_{\ell, p} \leq C_p \left(P\left[|\xi_n| > \frac{1}{2}\right]^\theta + P\left[|\xi'_n| > \frac{1}{2}\right]^\theta \right)$$

for $\theta < 1/p$. Here in order to obtain $P[|\xi'_n| > 2^{-1}] = o(r_n^k)$ for any $k > 0$, we can apply the truncation by ξ_n for nondegeneracy of σ_{Q_n} . After all, we obtain the desired result by Lemma 3. \square

Proof of Theorem 2. It holds that $P[r_n^{-c} |C_n - C_\infty| > 1] = O(r_n^{1+\kappa'})$ for any constant $c \in (2/3, 1)$ and any $\kappa' > 0$, due to the boundedness of $\{\check{C}_n\}$ in $L^{\infty-} = \cap_{p>1} L^p$. We take constants a and c so that $2/3 < 1-a < c < 1$. Let

$$\xi_n = 10^{-1} r_n^{-2c} |C_n - C_\infty|^2 + 2 \left[1 + 4 \Delta_{(M_n + W_\infty, F_\infty)} s_n^{-1} \right]^{-1}.$$

We only consider sufficiently large n . Then $[B2]_\ell$ and $[B3]$ are verified under $[B2']_\ell$ and $[B3']$. We take a sufficiently large $\theta \in (0, 1)$ depending on $\kappa \wedge \kappa'$, and apply Theorem 1. Note that

$$\left\{ |\xi_n| > \frac{1}{2} \right\} \subset \left\{ r_n^{-c} |C_n - C_\infty| > 1 \right\} \cup \left\{ \Delta_{(M_n + W_\infty, F_\infty)} < s_n \right\}.$$

□

6 Expansion for the double stochastic integral

6.1 Kernel of the quadratic variation

Hereafter, let $\ell = \check{d} + 8$. In order to fix ideas, we will consider the kernel function $K^n(s, r)$ defined by

$$K^n(s, r) = r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \odot 1_{(t_{j-1}, s]}(r) \ddot{K}^n(r) \quad (24)$$

for $\dot{K}^n \in \mathbb{D}_{\ell+1, \infty}(H \otimes \mathbb{R}^d)$ and $\ddot{K}^n \in \mathbb{D}_{\ell+1, \infty}(H \otimes \mathbb{R}^r \otimes \mathbb{R}^d)$, where \odot stands for the tensor product with partial Hadamard product given the index $(i, \alpha) \in \{1, \dots, d\} \times \{1, \dots, r\}$. Here H is identified with $L^2([0, 1]; \mathbb{R}^r)$, and the partial Hadamard product (entrywise product) given an index $\lambda \in \Lambda$ is defined as follows: for $a = (a_{\lambda j})_{\lambda \in \Lambda, j \in \mathcal{J}} \in \mathbb{R}^\Lambda \otimes \mathbb{R}^\mathcal{J}$ and $b = (b_{\lambda k})_{\lambda \in \Lambda, k \in \mathcal{K}} \in \mathbb{R}^\Lambda \otimes \mathbb{R}^\mathcal{K}$, $a \odot b \in \mathbb{R}^\Lambda \otimes \mathbb{R}^\mathcal{J} \otimes \mathbb{R}^\mathcal{K}$ is given by

$$a \odot b = (a_{\lambda j} b_{\lambda k})_{\lambda \in \Lambda, j \in \mathcal{J}, k \in \mathcal{K}}.$$

When \mathcal{J} and \mathcal{K} are one-point sets, \odot is a usual Hadamard product \odot . The sequence $\{t_j\}_{j=0,1,\dots,\bar{j}^n}$ ($n \in \mathbb{N}$) is a triangular array of numbers such that $t_j = t_j^n$ depending on n , $0 = t_0 < t_1 < \dots < t_{\bar{j}^n} = 1$, $\max_j |I_j| = o(r_n)$ and the sequence of measures

$$\mu^n = r_n^{-2} \sum_j |I_j|^2 \delta_{t_{j-1}} \rightarrow \mu \quad (25)$$

weakly for some measure μ on $[0, 1]$ with a bounded derivative, where $I_j = (t_{j-1}, t_j]$. Suppose that \dot{K}^n and \ddot{K}^n are progressively measurable. More strongly, we assume the **strong predictability condition** that $\dot{K}^n(s)$ is $\mathcal{F}_{t_{j-1}}$ -measurable for $s \in (t_{j-1}, t_j]$.⁹ We write

$$\bar{K}^n(s, r) = \dot{K}^n(s) \odot \ddot{K}^n(r).$$

Corresponding to the representation (24), we consider M_t^n given by

$$M_t^n = r_n^{-1} \sum_j \int_{t_{j-1} \wedge t}^{t_j \wedge t} \dot{K}^n(s) \odot \left(\int_{t_{j-1}}^s \ddot{K}^n(r) dw_r \right) dw_s,$$

where w is the canonical process on \mathbb{W} , extended naturally to $\bar{\Omega}$.

In this case,

$$C_n = \text{Tr}^o \int_0^1 r_n^{-2} \sum_j 1_{I_j}(s) \left(\dot{K}^n(s) \odot \int_{t_{j-1}}^s \ddot{K}^n(r) dw_r \right)^{\otimes 2} ds$$

and it will turn out that the in-p limit is

$$C_\infty = \frac{1}{2} \text{Tr}^* \int_0^1 \bar{K}^\infty(t, t)^{\otimes 2} \mu(dt), \quad (26)$$

where Tr^o and Tr^* denote the traces of the element of $L(\mathbb{R}^r; \mathbb{R}^r)$ and $L(\mathbb{R}^r \otimes \mathbb{R}^r; \mathbb{R}^r \otimes \mathbb{R}^r)$, respectively.

Let $\Delta = \{(s, r); 0 \leq r \leq s \leq 1\}$. Let $\Delta^n = \cup_j \{(s, r); t_{j-1} \leq r \leq s \leq t_j\}$. Let $\delta_n = r_n^{-2} \sum_j |I_j|^2 = O(1)$ as assumption.

⁹Obviously this condition is satisfied if one replaces $\dot{K}^n(s)$ by $\dot{K}^n(t_{j-1})$ in the representation of the kernel function $K^n(s, r)$. In examples, it will turn out that this replacement does not cause any practical difficulty.

Lemma 4. Suppose that K^n satisfy the following conditions.¹⁰

(i) $\sup_{n \in \mathbb{N}} \sup_{(s,r) \in \Delta^n} \|K^n(s,r)\|_p < \infty$ for every $p > 1$.

(ii) For every $p > 1$,

$$\sup_j \sup_{(s,r) \in \Delta^n, s \in I_j} \|\bar{K}^n(s,r) - \bar{K}^n(t_{j-1}, t_{j-1})\|_p \rightarrow 0$$

as $n \rightarrow \infty$.

(iii) There is a continuous process $\bar{K}^\infty(t, t)$ such that

$$\sup_j \sup_{t \in I_j} \|\bar{K}^n(t_{j-1}, t_{j-1}) - \bar{K}^\infty(t, t)\|_1 \rightarrow 0$$

as $n \rightarrow \infty$.¹¹

Then (26) holds true.

Proof. Condition (ii) together with the convergence of μ^n ensures

$$\begin{aligned} C_n &= \text{Tr}^o \int_0^1 r_n^{-2} \sum_j 1_{I_j}(s) \left(\dot{K}^n(s) \otimes \int_{t_{j-1}}^s \ddot{K}^n(r) dw_r \right)^{\otimes 2} ds \\ &= \mathfrak{C}_n + o_p(\delta_n), \end{aligned}$$

where

$$\mathfrak{C}_n = \text{Tr}^o \int_0^1 r_n^{-2} \sum_j 1_{I_j}(s) \left(\dot{K}^n(t_{j-1}) \otimes \int_{t_{j-1}}^s \ddot{K}^n(t_{j-1}) dw_r \right)^{\otimes 2} ds$$

and $o_p(\delta_n)$ denotes a sequence of matrices of indicated order. We see

$$\begin{aligned} \mathfrak{C}_n &= r_n^{-2} \sum_j \text{Tr}^o \text{Tr}^* \left\{ \bar{K}^n(t_{j-1}, t_{j-1})^{\otimes 2} \otimes \int_{t_{j-1}}^{t_j} \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^{\otimes 2} - \int_{t_{j-1}}^s \mathbf{I}_r dr \right\} ds \right\} \\ &\quad + r_n^{-2} \sum_j \text{Tr}^o \text{Tr}^* \left\{ \bar{K}^n(t_{j-1}, t_{j-1})^{\otimes 2} \otimes \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^s \mathbf{I}_r dr ds \right\}. \end{aligned} \quad (27)$$

The square of L^2 norm of the first term on the right-hand side is of the order

$$r_n^{-4} \sum_j |I_j|^4 \leq (r_n^{-1} \max_j |I_j|)^2 r_n^{-2} \sum_j |I_j|^2 \rightarrow 0$$

as $n \rightarrow \infty$. The second term converges in probability to the right-hand side of (26) by the L^p -equi-continuity of the random kernels. \square

Remark 3. It may seem that we can specify the behavior of \mathring{C}_n , however it is still related to the Itô expansion of the process $\dot{K}^n(s)$, so we have left this procedure to each individual case. Later we will consider the situation where $\dot{K}^n(s) = \dot{K}^n(t_{j-1})$ for $s \in (t_{j-1}, t_j]$. It is an easily tractable case in the above sense. We will do with it there.

Let $\mathbb{I}_s = -\frac{1}{2}(C_s^\infty - C_1^\infty)$, with C_s^∞ being the limit of C_s^n .

¹⁰Clearly, in order to show the result, the assumption is much stronger than necessary.

¹¹The reader may use " $\bar{K}^\infty(t)$ " for $\bar{K}^\infty(t, t)$ while we prefer the latter notation.

[A1] (i) $\bar{K}^n \in \mathbb{D}_{\ell+1}(H \otimes H \otimes \mathbb{R}^d)$ and a representation density of each derivative admits

$$ess. \sup_{\substack{r_1, \dots, r_k \in (0,1), \\ (s,r) \in \Delta^n, n \in \mathbb{N}}} \left\| D_{r_1, \dots, r_k} \bar{K}^n(s, r) \right\|_p < \infty$$

for every $p \in (1, \infty)$ and $k \leq \ell + 1$.

(ii) For every $\eta > 0$ and $p \in (1, \infty)$,

$$\sup_{s \in (0,1), e \in S^{d-1}} \left\| \left[\frac{\mathbb{I}_s[e^{\otimes 2}]}{(1-s)^{1+\eta}} \right]^{-1} \right\|_p < \infty.$$

(iii) $\sup_{s \in [0,1], n \in \mathbb{N}} |C_s^n - C_s^\infty| \leq r_n^{2q}$ whenever $|\xi_n| \leq 1$.

(iv) For every $p > 1$,

$$\sup_j \sup_{\substack{(s,r) \in \Delta^n \\ s \in I_j}} \|\bar{K}^n(s, r) - \bar{K}^n(t_{j-1}, t_{j-1})\|_{\ell, p} = O(r_n^{2q})$$

and

$$\sup_j \sup_{t \in I_j} \|\bar{K}^n(t_{j-1}, t_{j-1}) - \bar{K}^\infty(t, t)\|_{\ell, p} = O(r_n^{2q})$$

as $n \rightarrow \infty$.

Remark 4. (i) We note that $2q = 1 - a \in (\frac{2}{3}, 1)$ for (iii); in typical cases, $r_n^{2q} = n^{-q} > n^{-\frac{1}{2}}$. It is essentially possible to remove (iii) by redefining ξ_n , as it will be done under another set of conditions later. However, we here keep (iii) because it shows a role of ξ_n and making such ξ_n in each case is rather routine.

(ii) Under [A1] (i), for every $p \in (1, \infty)$ and $k \leq \ell$,

$$ess. \sup_{r_1, \dots, r_k, s \in (0,1)} \left\| \frac{D_{r_1, \dots, r_k} \mathbb{I}_s}{1-s} \right\|_p < \infty.$$

(iii) As for [A1] (ii), in order to do with $\exp(\frac{1}{2}(C_s^\infty - C_1^\infty)[u^{\otimes 2}])$ for s near 1, we use the nondegeneracy of the derivative of C_s^∞ in s , or a large deviation argument.

(iv) The nondegeneracy $\det C_\infty^{-1} \in \cap_{p \geq 2} L^p$ follows from [A1] (ii). Indeed, it implies

$$\sup_{e \in S^{d-1}} P[C_\infty[e^{\otimes 2}] < \epsilon] \leq \epsilon^p \sup_{e \in S^{d-1}} \|\mathbb{I}_0[e^{\otimes 2}]^{-1}\|_p^p \leq C_p \epsilon^p \quad (\epsilon > 0)$$

for some constant C_p for every $p > 1$. Then the desired inequality is obtained; see e.g. Lemma 2.3.1 of Nualart [15].

We assume:

[A2]^b (i) $F_\infty \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^{d_1})$ and $W_\infty \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^d)$.

(ii) $F_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^{d_1})$, $W_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^d)$, $N_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^d)$ and $\xi_n \in \mathbb{D}_{\ell, \infty}(\mathbb{R})$. Moreover,

$$\sup_{n \in \mathbb{N}} \left\{ \|\overset{\circ}{C}_n\|_{\ell, p} + \|\overset{\circ}{W}_n\|_{\ell+1, p} + \|\overset{\circ}{F}_n\|_{\ell+1, p} + \|N_n\|_{\ell+1, p} + \|\xi_n\|_{\ell, p} \right\} < \infty$$

for every $p \geq 2$.

(iii) $\underline{d} \in \mathcal{S}(\ell_*, 2, 1)$.

(iv) $(M^n, N_n, \overset{\circ}{C}_n, \overset{\circ}{W}_n, \overset{\circ}{F}_n) \rightarrow^{d_s(\mathcal{F})} (M^\infty, N_\infty, \overset{\circ}{C}_\infty, \overset{\circ}{W}_\infty, \overset{\circ}{F}_\infty)$.

(v) For $G = W_\infty$ and F_∞ ,

$$\text{ess.} \sup_{r_1, \dots, r_k \in (0,1)} \|D_{r_1, \dots, r_k} G\|_p < \infty$$

for every $p \in [2, \infty)$ and $k \leq \ell + 1$. Moreover, $r \mapsto D_r G$ and $(r, s) \mapsto D_{r,s} G$ ($r \leq s$) are continuous a.s.

Remark 5. Under the assumptions, $\bar{\sigma} \in \mathcal{S}(\ell, \mathbf{m}, \mathbf{n})$ and $\mathcal{F}_{(u,v)}^{-1}[\tilde{\Phi}^{2,0}](z, x)$ will have the random symbol $\bar{\sigma}$, and $\mathcal{L}\{M_t^\infty | \mathcal{F}\} = N_d(0, C_t^\infty)$. [A2]^h (iv) is a condition for the joint convergence since we have not specified the random variables other than M^n . We set (iii) by a similar reason.

The nondegeneracy of $(M_t^n + W_\infty, F_\infty)$ will be necessary.

[A3]^h (i) There exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, 1]$ with $\sup_n t_n < 1$ such that the family $\{(M_t^n + W_\infty, F_\infty)\}_{t \geq t_n, n \in \mathbb{N}}$ is uniformly nondegenerate under the truncation by ξ_n , namely,

$$\sup_{t \geq t_n, n \in \mathbb{N}} E[1_{\{|\xi_n| \leq 1\}} (\det \sigma_{(M_t^n + W_\infty, F_\infty)})^{-p}] < \infty$$

for every $p > 1$,

(ii) $\lim_{n \rightarrow \infty} P[|\xi_n| \leq \frac{1}{2}] = 1$.

Remark 6. We removed the condition “ $|C_n - C_\infty| > r_n^{1-a}$ implies $|\xi_n| \geq 1$, where $a \in (0, 1/3)$ is a constant” because we can modify the definition of ξ_n to satisfy this condition. It is possible thanks to L^p -boundedness of \mathring{C}_n . As for t_n , typically $t_n = 1/2$.

6.2 Anticipative random symbol and estimates of Fourier transforms

We are working with [A1], [A2]^h and [A3]^h. We denote $A_j^n(s) = 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s)$ and $B_j^n(s, r) = 1_{\{t_{j-1} < r < s \leq t_j\}} \ddot{K}^n(r)$.

Let $A_j^n = (A_{j,\alpha}^{n,\lambda})_{\substack{\lambda=1,\dots,d \\ \alpha=1,\dots,r}}$ and $B_j^n = (B_{j,\alpha,\beta}^{n,\lambda})_{\substack{\lambda=1,\dots,d \\ \alpha,\beta=1,\dots,r}}$. The α -th entry of the density function $D_t F$ taking values in \mathbb{R}^r will be denoted by $D_t^{(\alpha)} F$. We apply the IBP formula to obtain

$$\begin{aligned} E[L_1^n(u) \Psi_\infty(u, v) \psi_n] &= r_n^{-1} \sum_j iE \left[D^* \left(e^n(u) A_j^n(\cdot) \odot D^* B_j^n(\cdot) \right) [u] \Psi_\infty(u, v) \psi_n \right] \\ &= r_n^{-1} \sum_j iE \left[\left\langle \left(e^n(u) A_j^n(\cdot) \odot D^* B_j^n(\cdot) \right) [u], D(\Psi_\infty(u, v) \psi_n) \right\rangle_H \right] \\ &= ir_n^{-1} \sum_j \sum_{\lambda, \alpha} \int_0^1 E \left[u^\lambda e_s^n(u) A_{j,\alpha}^{n,\lambda}(s) D^* B_{j,\alpha,\cdot}^{n,\lambda}(s) D_s^{(\alpha)} (\Psi_\infty(u, v) \psi_n) \right] ds \\ &= ir_n^{-1} \sum_j \sum_{\lambda, \alpha} \int_0^1 E \left[u^\lambda \left\langle B_{j,\alpha,\cdot}^{n,\lambda}(s, \cdot), D \left\{ e_s^n(u) A_{j,\alpha}^{n,\lambda}(s) D_s^{(\alpha)} (\Psi_\infty(u, v) \psi_n) \right\} \right\rangle_H \right] ds \\ &= ir_n^{-1} \sum_j \sum_{\lambda, \alpha, \beta} \int_0^1 \int_0^1 E \left[u^\lambda B_{j,\alpha,\beta}^{n,\lambda}(s, r) D_r^{(\beta)} \left\{ e_s^n(u) A_{j,\alpha}^{n,\lambda}(s) D_s^{(\alpha)} (\Psi_\infty(u, v) \psi_n) \right\} \right] ds dr. \end{aligned}$$

Since \dot{K}^n is strongly predictable,

$$E[L_1^n(u) \Psi_\infty(u, v) \psi_n] = i \int_0^1 \int_0^1 E \left[\left\langle K^n(s, r) [u], D_r \left\{ e_s^n(u) D_s (\Psi_\infty(u, v) \psi_n) \right\} \right\rangle_{\mathbb{R}^r \otimes \mathbb{R}^r} \right] ds dr.$$

or more generally

$$\begin{aligned} & \mathfrak{D}^\alpha E [L_1^n(u) \Psi_\infty(u, v) \psi_n] \\ &= \sum_{\alpha_0 + \alpha_1 = \alpha} c_{\alpha_0, \alpha_1} i \int_0^1 \int_0^1 E \left[\left\langle \mathfrak{D}^{\alpha_0} K^n(s, r)[u], \mathfrak{D}^{\alpha_1} D_r \left\{ e_s^n(u) D_s (\Psi_\infty(u, v) \psi_n) \right\} \right\rangle_{\mathbb{R}^r \otimes \mathbb{R}^r} \right] ds dr \end{aligned}$$

with some constants c_{α_0, α_1} appearing in Leibniz's rule for \mathfrak{D}^α . We will find a representation of the random symbol $\bar{\sigma}$ associated to $\mathcal{F}_{(u, v)}^{-1}[\Phi^{2, 0}](z, x)$ and verify the convergence (9).

The density function appearing here is

$$\begin{aligned} & D_r \left\{ e_s^n(u) D_s (\Psi_\infty(u, v) \psi_n) \right\} \\ &= D_r \left\{ e_s^n(u) D_s \left(\exp \left\{ i W_\infty[u] - \frac{1}{2} C_\infty[u^{\otimes 2}] + i F_\infty[v] \right\} \psi_n \right) \right\} \\ &= e_s^n(u) \Psi_\infty(u, v) \psi_n \left(i D_r M_s^n[u] + \frac{1}{2} D_r C_s^n[u^{\otimes 2}] \right) \otimes \left(i D_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + i D_s F_\infty[v] \right) \\ &\quad + e_s^n(u) \Psi_\infty(u, v) \psi_n \left(i D_r W_\infty[u] - \frac{1}{2} D_r C_\infty[u^{\otimes 2}] + i D_r F_\infty[v] \right) \otimes \left(i D_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + i D_s F_\infty[v] \right) \\ &\quad + e_s^n(u) \Psi_\infty(u, v) \psi_n \left(i D_r D_s W_\infty[u] - \frac{1}{2} D_r D_s C_\infty[u^{\otimes 2}] + i D_r D_s F_\infty[v] \right) \\ &\quad + e_s^n(u) \Psi_\infty(u, v) D_r \psi_n \otimes \left(i D_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + i D_s F_\infty[v] \right) \\ &\quad + D_r \left\{ e_s^n(u) \Psi_\infty(u, v) D_s \psi_n \right\}. \end{aligned}$$

Moreover, the Leibniz type formulas for $\mathfrak{D}^{\alpha_2} D_r \left\{ e_s^n(u) D_s (\Psi_\infty(u, v) \psi_n) \right\}$ remain in force if one applies the Leibniz rule to each term on the right-hand side of the above equality. It will be observed that the function $|\mathfrak{D}^\alpha E [L_1^n(u) \Psi_\infty(u, v) \psi_n]|$ is dominated by a polynomial in (u, v) at most fifth-order under a restricted range of (u, v) by means of the truncation with ψ_n .

We introduce an $\mathbb{R}^r \otimes \mathbb{R}^r$ -valued random symbol

$$\begin{aligned} \sigma_{s, r}(iu, iv) &= \frac{1}{2} D_r C_s^n[u^{\otimes 2}] \otimes \left(i D_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + i D_s F_\infty[v] \right) \\ &\quad + \left(i D_r W_\infty[u] - \frac{1}{2} D_r C_\infty[u^{\otimes 2}] + i D_r F_\infty[v] \right) \otimes \left(i D_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + i D_s F_\infty[v] \right) \\ &\quad + \left(i D_r D_s W_\infty[u] - \frac{1}{2} D_r D_s C_\infty[u^{\otimes 2}] + i D_r D_s F_\infty[v] \right). \end{aligned}$$

6.2.1 The terms involving $D_r M_s^n$

We have

$$\begin{aligned} D_r M_s^n &= r_n^{-1} \sum_j \int_{t_{j-1} \wedge s}^{t_j \wedge s} D_r \dot{K}^n(s_1) \otimes \left(\int_{t_{j-1}}^{s_1} \ddot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \\ &\quad + r_n^{-1} \sum_j \int_{t_{j-1} \wedge s}^{t_j \wedge s} \dot{K}^n(s_1) \otimes \left(\int_{t_{j-1}}^{s_1} D_r \dot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \\ &\quad + r_n^{-1} \sum_j \int_{t_{j-1} \wedge s}^{t_j \wedge s} \dot{K}^n(s_1) \otimes 1_{(t_{j-1}, s_1]}(r) \ddot{K}^n(r) dw_{s_1} \\ &\quad + r_n^{-1} \sum_j 1_{(t_{j-1} \wedge s, t_j \wedge s]}(r) \dot{K}^n(r) \otimes \int_{t_{j-1}}^r \ddot{K}^n(s_1) dw_{s_1}. \end{aligned}$$

With this representation, we see

$$\begin{aligned}
\int_0^1 K^n(s, r) \otimes D_r M_s^n dr &= \int_0^1 r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \otimes 1_{(t_{j-1}, s]}(r) \ddot{K}^n(r) \\
&\otimes \left\{ r_n^{-1} \sum_{j'} \int_{t_{j'-1} \wedge s}^{t_{j'} \wedge s} D_r \dot{K}^n(s_1) \otimes \left(\int_{t_{j'-1}}^{s_1} \ddot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \right. \\
&+ r_n^{-1} \sum_{j'} \int_{t_{j'-1} \wedge s}^{t_{j'} \wedge s} \dot{K}^n(s_1) \otimes \left(\int_{t_{j'-1}}^{s_1} D_r \ddot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \\
&+ r_n^{-1} \sum_{j'} \int_{t_{j'-1} \wedge s}^{t_{j'} \wedge s} \dot{K}^n(s_1) 1_{(t_{j'-1}, s_1]}(r) \otimes \ddot{K}^n(r) dw_{s_1} \\
&\left. + r_n^{-1} \sum_{j'} 1_{(t_{j'-1} \wedge s, t_{j'} \wedge s]}(r) \dot{K}^n(r) \otimes \int_{t_{j'-1}}^r \ddot{K}^n(s_1) dw_{s_1} \right\} dr.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_0^1 K^n(s, r) \otimes D_r M_s^n dr \\
&= r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \otimes r_n^{-1} \int_{t_{j-1}}^s \left\{ \int_{t_{j-1} \wedge s}^{t_j \wedge s} \ddot{K}^n(r) \otimes D_r \dot{K}^n(s_1) \otimes \left(\int_{t_{j-1}}^{s_1} \ddot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \right. \\
&\quad + \ddot{K}^n(r) \otimes \int_{t_{j-1} \wedge s}^{t_j \wedge s} \dot{K}^n(s_1) \otimes \left(\int_{t_{j-1}}^{s_1} D_r \ddot{K}^n(s_2) dw_{s_2} \right) dw_{s_1} \\
&\quad + \ddot{K}^n(r) \otimes \int_{t_{j-1} \wedge s}^{t_j \wedge s} \dot{K}^n(s_1) \otimes 1_{(t_{j-1}, s_1]}(r) \ddot{K}^n(r) dw_{s_1} \\
&\quad \left. + 1_{(t_{j-1} \wedge s, t_j \wedge s]}(r) \dot{K}^n(r) \otimes \dot{K}^n(r) \otimes \int_{t_{j-1}}^r \ddot{K}^n(s_1) dw_{s_1} \right\} dr.
\end{aligned}$$

Here we note that for terms not to vanish, it is necessary that

$$r \leq s_1 \leq t_{j'} \quad \text{for } D_r \dot{K}^n(s_1) \neq 0 \quad (28)$$

$$r \leq s_2 \leq s_1 \leq t_{j'} \quad \text{for } D_r \ddot{K}^n(s_2) \neq 0 \quad (29)$$

$$t_{j-1} < r \leq s \quad (30)$$

$$t_{j-1} < s \leq t_j \quad (31)$$

$$t_{j'-1} < s \quad (32)$$

In particular, $t_{j-1} < t_{j'}$ from [(28) or (29)] and (30), and $t_{j'-1} < t_j$ from (31) and (32), so that $j' = j$. Similar argument is valid for the last two terms to neglect off-diagonal elements for $j \neq j'$.

We will assume that

$$r_n^{-8} \sum_j |I_j|^5 = O(1) \quad (33)$$

as $n \rightarrow \infty$. Then, applying Jensen's inequality, we have

$$r_n^{(1)} := r_n^{-2} \sum_j |I_j|^{\frac{5}{2}} = O(r_n) \quad (34)$$

and

$$r_n^{(2)} := r_n^{-2} \sum_j |I_j|^3 = O(r_n^2), \quad (35)$$

moreover, $r_n^{(2)} \leq O(r_n^{\frac{1}{2}} r_n^{(1)})$ since $\max_j |I_j| = O(r_n)$ by assumption.

Let

$$Q_s = iD_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + iD_s F_\infty[v].$$

Without scaling by r_n^{-1} ,

$$\begin{aligned} \mathcal{I}_n &:= i \int_0^1 \int_0^1 E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n iD_r M_s^n[u] \otimes Q_s \right\rangle_{\mathbb{R}^r \otimes \mathbb{R}^r} \right] ds dr \\ &= -\text{Tr}^* E \left[\int_0^1 ds e_s^n(u) \Psi_\infty(u, v) \psi_n r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \right. \\ &\quad \otimes r_n^{-1} \int_{t_{j-1}}^s \dot{K}^n(r) \otimes \left\{ \int_{t_{j-1} \wedge s}^{t_j \wedge s} \dot{K}^n(s_1) 1_{(t_{j-1}, s_1]}(r) \otimes \dot{K}^n(r) dw_{s_1} \right. \\ &\quad \left. \left. + 1_{(t_{j-1} \wedge s, t_j \wedge s]}(r) \dot{K}^n(r) \otimes \int_{t_{j-1}}^r \dot{K}^n(s_1) dw_{s_1} \right\} dr \otimes Q_s \right] [u^{\otimes 2}] + O(r_n^{(2)}) \\ &= -\text{Tr}^* \int_0^1 E \left[e_s^n(u) \Psi_\infty(u, v) \psi_n r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \right. \\ &\quad \cdot r_n^{-1} \int_{t_{j-1}}^s \bar{K}^n(s, r) \otimes \left\{ \int_{t_{j-1} \wedge s}^{t_j \wedge s} 1_{(t_{j-1}, s_1]}(r) \bar{K}^n(s_1, r) dw_{s_1} \right. \\ &\quad \left. \left. + 1_{(t_{j-1} \wedge s, t_j \wedge s]}(r) \dot{K}^n(r) \otimes \int_{t_{j-1}}^r \dot{K}^n(s_1) dw_{s_1} \right\} \otimes Q_s dr \right] [u^{\otimes 2}] ds + O(r_n^{(2)}) \\ &= -\text{Tr}^* E \left[\Psi_\infty(u, v) \psi_n \sum_j \xi_j^n(u) \otimes Q_{t_{j-1}} \right] [u^{\otimes 2}] + o(r_n^{(1)}), \end{aligned} \quad (36)$$

where

$$\begin{aligned} \xi_j^n(u) &= e_{t_{j-1}}^n(u) \bar{K}^n(t_{j-1}, t_{j-1}) \otimes \bar{K}^n(t_{j-1}, t_{j-1}) r_n^{-2} \left\{ \int_{t_{j-1}}^{t_j} (t_j - s_1)(s_1 - t_{j-1}) dw_{s_1} \right. \\ &\quad \left. + \int_{t_{j-1}}^{t_j} \frac{(t_j - s_1)^2}{2} dw_{s_1} \right\}. \end{aligned}$$

The last equality in the expression of \mathcal{I}_n is by straightforward L^p -estimate with triangular inequality and L^p -continuity of the process Q_s ; note that Q_s may be anticipative and we do not assume the predictability of $Q_{t_{j-1}}$, which has no more meaning than approximation to Q_s .

Let

$$\begin{aligned} I^n(u, v) &= r_n^{-1} i \int_0^1 \int_0^1 E \left[\left\langle \mathfrak{K}^{\alpha_0} K^n(s, r)[u], \mathfrak{K}^{\alpha_1} \left\{ e_s^n(u) \Psi_\infty(u, v) \psi_n iD_r M_s^n[u] \right. \right. \right. \\ &\quad \left. \left. \otimes \left(iD_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + iD_s F_\infty[v] \right) \right\} \right\rangle_{\mathbb{R}^r \otimes \mathbb{R}^r} \right] ds dr \end{aligned} \quad (37)$$

We will apply the integration-by-parts formula $\check{d} + 6$ times for $s \geq t_n$, by taking advantage of the nondegeneracy of $\{(M_s^n + W_\infty, F_\infty)\}_{s \geq t_n, n \in \mathbb{N}}$ when $|\xi_n| \leq 1$, and that of the decay of $\exp\{\frac{1}{2}(C_s^n - C_\infty)[u^{\otimes 2}]\}$ for $s < t_n$ as well as the integration-by-parts formula for F_∞ in order to obtain integrability in (u, v) of $I^n(u, v)$ of (37). We need several steps to achieve this plan.

(a) $(M_s^n + W_\infty, F_\infty)$ and $C_s^n - C_\infty$ appear in the factorization

$$\begin{aligned} e_s^n(u) \Psi_\infty(u, v) &= \exp \left(i M_s^n[u] + \frac{1}{2} (C_s^n - C_\infty)[u^{\otimes 2}] \right) \exp \left(i W_\infty[u] + i F_\infty[v] \right) \\ &= \mathbb{F}_s^n \mathbb{G}_s \mathbb{H}_s^n, \end{aligned}$$

where

$$\begin{aligned} \mathbb{F}_s^n &= \exp \left(i (M_s^n + W_\infty)[u] + i F_\infty[v] \right), \\ \mathbb{G}_s &= \exp \left(\frac{1}{2} (C_s^\infty - C_1^\infty)[u^{\otimes 2}] \right), \\ \mathbb{H}_s^n &= \exp \left(\frac{1}{2} (C_s^n - C_s^\infty)[u^{\otimes 2}] \right). \end{aligned}$$

(b) By the Leibniz rule, the density function $D_{r_1, \dots, r_k} \exp(-\mathbb{I}_s[u^{\otimes 2}])$ of the k th Malliavin derivative of $\exp(-\mathbb{I}_s[u^{\otimes 2}])$ is a linear combination of the terms of the form

$$\mathbb{A} = D_{r_1, \dots, r_{k_1}} \mathbb{I}_s[u^{\otimes 2}] \otimes \dots \otimes D_{r_{k_{m-1}+1}, \dots, r_{k_m}} \mathbb{I}_s[u^{\otimes 2}] \exp(-\mathbb{I}_s[u^{\otimes 2}]),$$

where $1 \leq k_1 < \dots < k_{m-1} < k_m = k$ and $1 \leq m \leq k$. We denote by ϵ a positive number and we will make its value as small as we want in the context. It is possible to choose such an ϵ because we only change its values finitely often. Let $u \in \mathbb{R}^d \setminus \{0\}$ and let $e_u = |u|^{-1}u$.

$$\begin{aligned} |\mathbb{A}| &\leq \frac{|D_{r_1, \dots, r_{k_1}} \mathbb{I}_s|}{1-s} \dots \frac{|D_{r_{k_{m-1}+1}, \dots, r_{k_m}} \mathbb{I}_s|}{1-s} \left[\frac{\mathbb{I}_s[e_u^{\otimes 2}]}{(1-s)^{1+\eta(\epsilon)}} \right]^{-m(1-\epsilon)} \\ &\quad \cdot \mathbb{I}_s[u^{\otimes 2}]^{m(1-\epsilon)} \exp(-\mathbb{I}_s[u^{\otimes 2}]) |u|^{2m\epsilon} \end{aligned}$$

for $\eta(\epsilon) = \epsilon(1-\epsilon)^{-1}$.

(c) For every m and $\epsilon \in (0, 1)$, $\mathbb{I}_s[u^{\otimes 2}]^{m(1-\epsilon)} \exp(-\mathbb{I}_s[u^{\otimes 2}])$ is bounded since \mathbb{I}_s is nonnegative-definite and $\sup_{x \geq 0} |x|^m e^{-x} < \infty$. This and the inequality in (b) together with [A1] (i)-(ii) imply

$$ess. \sup_{r_1, \dots, r_k, s \in (0, 1)} \|\mathbb{A}\|_p \leq C(p) |u|^{2m\epsilon}$$

for some constant $C(p)$.

(d) $(C_s^n - C_s^\infty)[u, u]$ is bounded in $u \in \Lambda_n^0$ whenever $\xi_n \leq 1$, due to [A1] (iii). Therefore, by [A1] (iv), $\{\|D^k \mathbb{H}_s^n 1_{\{\xi_n \leq 1\}}\|_p; s \in [0, 1], u \in \Lambda(d, q), n \in \mathbb{N}\}$ is bounded for every $p > 1$ for $k \leq \ell$. We notice that due to (33), the first term on the right-hand side of (27) is $O(r_n)$ in norms.

(e) The Sobolev norms of $\{D_r M_s^n\}_{(r, s) \in [0, 1]^2, n \in \mathbb{N}}$ are bounded.

(f) $\sup_{n \in \mathbb{N}} r_n^{-2} \int_0^1 \int_0^1 \sum_j 1_{(t_{j-1}, t_j]}(s) 1_{(t_{j-1}, s]}(r) ds dr < \infty$.

(g) In the estimation of \mathbb{A} in (b), we can re-estimate the factor $\mathbb{I}_s[u^{\otimes 2}]^{m(1-\epsilon)} \exp(-\mathbb{I}_s[u^{\otimes 2}])$ as

$$\mathbb{I}_s[u^{\otimes 2}]^{m(1-\epsilon)} \exp(-\mathbb{I}_s[u^{\otimes 2}]) \leq C(k) |u|^{-2k} (\mathbb{I}_{t_n}[e_u^{\otimes 2}])^{-k}$$

whenever $\xi_n \leq 1$, for any $k \in \mathbb{N}$. It will be applied for $s < t_n$.

(h) With (a), (c), (d), (e) and (f), we repeatedly apply the integration-by-parts formula based on $(M_s^n + W_\infty, F_\infty)$ at (36) for $s \geq t_n$. For $s < t_n$, we apply the integration-by-parts formula based on F_∞ (for all $v \in \mathbb{R}^{d_1}$) with the help of (d) and (g), to obtain

$$\sup_n \sup_{(u, v) \in \Lambda_n^0(\vec{d}, q)} |(u, v)|^{\vec{d}+2-\epsilon'} |I^n(u, v)| < \infty \quad (38)$$

for some $\epsilon' \in (0, 1)$. We note that the differentiation with respect to (u, v) in (37) does not change the estimate essentially. Note also that $I^n(u, v)$ is like a 4th-order polynomial in (u, v) in growth rate, and that $\ell - 4 - \epsilon' = \check{d} + 2 - \epsilon'$.¹²

Let

$$\begin{aligned} \mathcal{M}_t^n &= r_n^{-1} \sum_j e_{t_{j-1}}^n(u) \bar{K}^n(t_{j-1}, t_{j-1}) \\ &\quad \otimes \bar{K}^n(t_{j-1}, t_{j-1}) r_n^{-2} \left\{ \int_{t_{j-1} \wedge t}^{t_j \wedge t} (t_j - s_1)(s_1 - t_{j-1}) dw_{s_1} + \int_{t_{j-1} \wedge t}^{t_j \wedge t} \frac{(t_j - s_1)^2}{2} dw_{s_1} \right\}. \end{aligned}$$

Obviously, the Burkholder-Davis-Gundy inequality applied under (33) implies

$$\| \sup_{t \in [0, 1]} |\mathcal{M}_t^n| \|_p = O(r_n)$$

for every $p > 1$.

For $p > 1$ and $\epsilon > 0$, thanks to (34), we can find an increasing sequence T_k ($k = 0, 1, \dots, K$) in $[0, 1]$ such that $0 = T_0 < \dots < T_K = 1$ and

$$\left| r_n^{-1} E \left[\Psi_\infty(u, v) \psi_n \sum_j \xi_j^n(u) \otimes (Q_{T_{k(t_{j-1})}} - Q_{t_{j-1}}) \right] \right| < \epsilon$$

uniformly for large n , where $k(t_{j-1}) = \max\{k; T_k \leq t_{j-1}\}$ depending on j and n . Moreover,

$$\begin{aligned} & r_n^{-1} E \left[\Psi_\infty(u, v) \psi_n \sum_j \xi_j^n(u) \otimes Q_{T_{k(t_{j-1})}} \right] \\ &= \sum_{k=1}^K E \left[\Psi_\infty(u, v) \psi_n (\mathcal{M}_{T_k}^n - \mathcal{M}_{T_{k-1}}^n) \otimes Q_{T_{k-1}} \right] + o(1) \\ &= o(1) \end{aligned}$$

since the stable limit is centered. Here the first equality was due to C-tightness of the sequence \mathcal{M}^n .

Since ϵ is arbitrary, as a consequence of the above estimates, we obtain

$$i \int_0^1 \int_0^1 E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n i D_r M_s^n[u] \otimes Q_s \right\rangle_{\mathbb{R}^r \otimes \mathbb{R}^r} \right] ds dr = o(r_n)$$

as $n \rightarrow \infty$. Even in the case involving derivatives in (u, v) , following the same argument, we obtain

$$I^n(u, v) = o(1) \tag{39}$$

as $n \rightarrow \infty$ for every $(u, v) \in \mathbb{R}^{\check{d}}$.

6.2.2 The terms involving $D_r C_s^n$ and others

By definition,

$$\int_0^1 K^n(s, r) \otimes D_r C_s^n dr = r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \odot \int_0^1 1_{(t_{j-1}, s]}(r) \dot{K}^n(r) \otimes D_r C_s^n dr.$$

¹²Here we have an index larger than necessary. But it becomes $\check{d} + 1 - \epsilon'$ for some other terms.

With scaling by r_n^{-1} ,

$$\begin{aligned}
& r_n^{-1} i \int_0^1 \int_0^1 E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n \frac{1}{2} D_r C_s^n [u^{\otimes 2}] \otimes Q_s \right\rangle_{\mathbb{R}' \otimes \mathbb{R}'} \right] ds dr \\
&= r_n^{-1} i \operatorname{Tr}^* \int_0^1 E \left[e_s^n(u) \Psi_\infty(u, v) \psi_n r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) \dot{K}^n(s) \otimes \int_0^1 1_{(t_{j-1}, s]}(r) \ddot{K}^n(r) \otimes \frac{1}{2} D_r C_s^n dr \otimes Q_s \right] [u^{\otimes 3}] ds \\
&= r_n^{-2} i \operatorname{Tr}^* \int_0^1 \sum_j 1_{(t_{j-1}, t_j]}(s) (s - t_{j-1}) E \left[e_s^n(u) \Psi_\infty(u, v) \psi_n \dot{K}^n(s) \otimes \dot{K}^n(t_{j-1}) \otimes \frac{1}{2} D_{t_{j-1}} C_s^n \otimes Q_s \right] [u^{\otimes 3}] ds + o(1) \\
&= r_n^{-2} i \operatorname{Tr}^* \sum_j \frac{1}{2} |I_j|^2 E \left[e_{t_{j-1}}^n(u) \Psi_\infty(u, v) \psi_n \dot{K}^n(t_{j-1}) \otimes \ddot{K}^n(t_{j-1}) \otimes \frac{1}{2} D_{t_{j-1}} C_{t_{j-1}}^n \otimes Q_{t_{j-1}} \right] [u^{\otimes 3}] + o(1) \\
&= \frac{i}{2} \operatorname{Tr}^* \int_0^1 E \left[e_t^\infty(u) \Psi_\infty(u, v) \bar{K}^\infty(t, t) \otimes \frac{1}{2} D_t C_t^\infty \otimes Q_t \right] [u^{\otimes 3}] \mu(dt) + o(1),
\end{aligned} \tag{40}$$

where $D_t C_t^\infty = \lim_{s \uparrow t} D_s C_s^\infty$. For the last equality, the first integration in the last line can be approximated by that with respect to μ^n . If we realize the weak convergence of random variables including $e^n(u)$ taking values in the space of continuous functions with uniform norm as a.s. convergence, then the integrand of the last line can be approximated by $E[e_t^n(u) \Psi_\infty(u, v) \psi_n \bar{K}^n(t, t) \otimes \frac{1}{2} D_t C_t^n \otimes Q_t]$ uniformly in t with the help of the truncation.

By conditioning,

$$\begin{aligned}
& E \left[e_t^\infty(u) \Psi_\infty(u, v) \bar{K}^\infty(t, t) \otimes D_t C_t^\infty \otimes Q_t \right] \\
&= E \left[\Psi_\infty(u, v) \bar{K}^\infty(t, t) \otimes D_t C_t^\infty \otimes Q_t \right] \\
&= E \left[\exp \left\{ i W_\infty[u] - \frac{1}{2} C_\infty[u^{\otimes 2}] + i F_\infty[v] \right\} \bar{K}^\infty(t, t) \otimes D_t C_t^\infty \otimes Q_t \right] \\
&= E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + i W_\infty[u] + i F_\infty[v]) \phi(z; 0, C_\infty) dz \bar{K}^\infty(t, t) \otimes D_t C_t^\infty \otimes Q_t \right]
\end{aligned}$$

Therefore

$$\begin{aligned}
& r_n^{-1} i \int_0^1 \int_0^1 E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n \frac{1}{2} D_r C_s^n [u^{\otimes 2}] \otimes Q_s \right\rangle_{\mathbb{R}' \otimes \mathbb{R}'} \right] ds dr \\
&= \frac{i}{2} E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + i W_\infty[u] + i F_\infty[v]) \phi(z; 0, C_\infty) dz \operatorname{Tr}^* \int_0^1 \bar{K}^\infty(t, t) \otimes \frac{1}{2} D_t C_t^\infty \otimes Q_t \mu(dt) \right] [u^{\otimes 3}] + o(1).
\end{aligned}$$

Moreover, $D_t C_t^\infty = 0$ in this case.

We can do with the terms involving either $D_r W_\infty$, $D_r F_\infty$, $D_r D_s W_\infty$, $D_r D_s C_\infty$, or $D_r D_s F_\infty$ in the same way to obtain

$$\begin{aligned}
\tilde{\Phi}^{2,0}(u, v) &:= \lim_{n \rightarrow \infty} r_n^{-1} \Phi_n^{2,0}(u, v) \\
&= \lim_{n \rightarrow \infty} r_n^{-1} E [L_1^n(u) \Psi_\infty(u, v) \psi_n] \\
&= \frac{i}{2} E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + i W_\infty[u] + i F_\infty[v]) \phi(z; 0, C_\infty) dz \operatorname{Tr}^* \int_0^1 \bar{K}^\infty(t, t)[u] \otimes \sigma_{t,t}(iu, iv) \mu(dt) \right] \\
&= E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + i F_\infty[v]) \phi(z; W_\infty, C_\infty) dz \bar{\sigma}(iu, iv) \right],
\end{aligned} \tag{41}$$

where $\sigma_{t,t}(iu, iv) = \lim_{s \uparrow t} \sigma_{t,s}(iu, iv)$ and

$$\bar{\sigma}(iu, iv) = \frac{1}{2} \operatorname{Tr}^* \int_0^1 \bar{K}^\infty(t, t)[iu] \otimes \sigma_{t,t}(iu, iv) \mu(dt). \tag{42}$$

The random symbol $\sigma_{t,t}(iu, iv)$ can be expressed formally as

$$\begin{aligned}\sigma_{t,t}(iu, iv) &= \left(iD_t W_\infty[u] - \frac{1}{2} D_t C_\infty[u^{\otimes 2}] + iD_t F_\infty[v] \right) \otimes \left(iD_t W_\infty[u] - \frac{1}{2} D_t C_\infty[u^{\otimes 2}] + iD_t F_\infty[v] \right) \\ &\quad + \left(iD_t D_t W_\infty[u] - \frac{1}{2} D_t D_t C_\infty[u^{\otimes 2}] + iD_t D_t F_\infty[v] \right).\end{aligned}$$

By tracing the derivation of the above limit with the Leibniz rule, we obtain in a similar manner

$$\begin{aligned}\tilde{\Phi}^{2,\alpha}(u, v) &:= \lim_{n \rightarrow \infty} r_n^{-1} \mathfrak{D}^\alpha \Phi_n^{2,0}(u, v) \\ &= \lim_{n \rightarrow \infty} r_n^{-1} \mathfrak{D}^\alpha E[L_1^n(u) \Psi_\infty(u, v) \psi_n] \\ &= \frac{i}{2} \mathfrak{D}^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iW_\infty[u] + iF_\infty[v]) \phi(z; 0, C_\infty) dz \operatorname{Tr}^* \int_0^1 \bar{K}^\infty(t, t)[u] \otimes \sigma_{t,t}(iu, iv) dt \right] \\ &= \mathfrak{D}^\alpha E \left[\int_{\mathbb{R}^d} \exp(iu \cdot z + iF_\infty[v]) \phi(z; W_\infty, C_\infty) dz \bar{\sigma}(iu, iv) \right].\end{aligned}$$

Applying the integration-by-parts formula at (40) or its derivatives in the same way as we reached (38), we obtain

$$\sup_n \sup_{(u,v) \in \Lambda_n^0(d,q)} |(u, v)|^{d+1-\epsilon'} r_n^{-1} |\Phi_n^{2,\alpha}(u, v)| < \infty. \quad (43)$$

6.3 Asymptotic expansion of the double stochastic integral

We are now on the point of presenting our results with the aid of the preceding subsection. The density p_n is given by (11), (10), (8) and (42).

Theorem 3. *Suppose that Conditions [A1], [A2][‡] and [A3][‡] are fulfilled. Let $M, \gamma \in (0, \infty)$ and $\theta \in (0, 1)$ be arbitrary numbers. Then, for some constant $C_1 = C(M, \gamma, \theta)$, (13) holds as $n \rightarrow \infty$.*

Proof. Inequality (12) has been verified in the present situation by (43). We can verify [B1], [B2]_ℓ, [B3] and [B4]_{ℓ, m, n} for $(\mathbf{m}, \mathbf{n}) = (5, 2)$ to apply Theorem 1 (b). [Recall that now ℓ is different from “ℓ” in Theorem 1.] We apply [10] for [B1] (ii), however we still need the joint convergence assumption [A2][‡] (iv). \square

We shall present a version of Theorem 3.

[A1]^b Conditions in [A1] hold except for (iii).

$s_n : \Omega \rightarrow \mathbb{R}$ is a positive functional.

[A2] Condition [A2][‡] holds, replacing its (ii) by

(ii) $F_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^{d_1})$, $W_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^d)$, $N_n \in \mathbb{D}_{\ell+1, \infty}(\mathbb{R}^d)$ and $s_n \in \mathbb{D}_{\ell, \infty}(\mathbb{R})$. Moreover,

$$\sup_{n \in \mathbb{N}} \left\{ \|\overset{\circ}{C}_n\|_{\ell, p} + \|\overset{\circ}{W}_n\|_{\ell+1, p} + \|\overset{\circ}{F}_n\|_{\ell+1, p} + \|N_n\|_{\ell+1, p} + \|s_n\|_{\ell, p} \right\} < \infty$$

for every $p \geq 2$.

The nondegeneracy of $(M_t^n + W_\infty, F_\infty)$ will be necessary.

[A3] (i) There exist a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, 1]$ with $\sup_n t_n < 1$ such that

$$\sup_{t \geq t_n} P[\det \sigma_{(M_t^n + W_\infty, F_\infty)} < s_n] = O(r_n^\nu) \text{ as } n \rightarrow \infty \text{ for some } \nu > \ell/3.$$

(ii) For every $p \geq 2$, $\limsup_{n \rightarrow \infty} E[s_n^{-p}] < \infty$.

Theorem 4. Suppose that Conditions [A1]^b, [A2] and [A3] are fulfilled. Then for any positive numbers M and γ ,

$$\sup_{f \in \mathcal{E}(M, \gamma)} \Delta_n(f) = o(r_n)$$

as $n \rightarrow \infty$.

Proof. As in the proof of Theorem 3, we will verify [B1], [B2] _{ℓ'} , [B3] and [B4] _{$\ell', \mathbf{m}, \mathbf{n}$} for $(\ell', \mathbf{m}, \mathbf{n}) = (\check{d} + 3, 5, 2)$ to apply Theorem 1 (b). Define ξ_n by

$$\begin{aligned} \xi_n &= 10^{-1} r_n^{-2c} |C_n - C_\infty|^2 + 2 \left[1 + 4\Delta_{(M_n + W_\infty, F_\infty)} s_n^{-1} \right]^{-1} \\ &\quad + L^* \int_{[0,1]^2} \left(\frac{|C_t^n - C_t^\infty - C_s^n + C_s^\infty| r_n^{-2q}}{|t-s|^{3/8}} \right)^8 dt ds, \end{aligned}$$

where c is a constant given in the proof of Theorem 2 and L^* is a sufficiently large constant. Here we can choose a number q that is smaller than the given q . Now the rest is to verify (12) for “ ℓ ” = $\check{d} + 3$, that is, (43). There were two steps to reach (43): (38) and the argument just before (43) concerning (40). The reasoning is quite the same in those cases, so we will show (38). However we need to do it under [A1]^b, [A2] and [A3] this time, not under [A1], [A2]^b and [A3]^b. Obviously we have [A2]^b. Condition [A1] (iii) is satisfied due to the definition of the above ξ_n with suitable L^* .

In order to estimate (36) once again under the present assumptions, we can follow (a)-(g) in Section 6.2.1. Let

$$\psi_{n,s} = \psi \left(2 \left[1 + 4\Delta_{(M_s^n + W_\infty, F_\infty)} s_n^{-1} \right]^{-1} \right)$$

for $s \geq t_n$. Then the integrand of (36) is decomposed as

$$\begin{aligned} & E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n i D_r M_s^n[u] \otimes Q_s \right\rangle_{\mathbb{R}' \otimes \mathbb{R}'} \right] \\ &= E \left[\left\langle K^n(s, r)[u], e_s^n(u) \Psi_\infty(u, v) \psi_n \psi_{n,s} i D_r M_s^n[u] \otimes Q_s \right\rangle_{\mathbb{R}' \otimes \mathbb{R}'} \right] + R_n(s, r) \end{aligned}$$

with

$$|R_n(s, r)| \leq C(p) r_n^{-\mathfrak{k}q} \sup_{s'} \|1 - \psi_{n,s'}\|_p r_n^{-1} \sum_j 1_{(t_{j-1}, t_j]}(s) 1_{(t_{j-1}, s]}(r)$$

for all n, s and restricted (u, v) , where $\mathfrak{k} = 4$ and $C(p)$ is a constant independent of them but on $p \in (1, \infty)$. Taking a small $q > 1/3$, we have $\nu - (\check{d} + 1 + \mathfrak{k})q > 0$ (even for $\mathfrak{k} = 5$). We can apply the integration-by-parts formula repeatedly as in (h) of Section 6.2.1 with truncation by $\psi_n \psi_{n,s}$ instead of ψ_n , and then (38) follows from [A3](i) by choosing a small $p > 1$. It was the estimate for the terms concerning $D_r M_s^n$. We can do the same kind estimate for the terms involving $D_r C_s^n$ and others to finally obtain (43) while $\mathfrak{k} = 5$ in this case. \square

7 Quadratic form of a Wiener process

7.1 Asymptotic expansion

The quadratic form of the increments of a diffusion process with a strongly predictable kernel plays a central role in the inference for diffusion coefficients. The case of the Wiener process shows what is most essential in

the analysis. Let $a \in C_{\uparrow}^{\infty}(\mathbb{R})$, the set of smooth functions with all derivatives at most polynomial growth. We write $1_j = 1_{(t_{j-1}, t_j]}$, $t_j = j/n$. Let

$$M_t^n = \sqrt{n} \sum_j 2a(w_{t_{j-1}}) \int_{t_{j-1} \wedge t}^{t_j \wedge t} \int_{t_{j-1}}^s dw_r dw_s.$$

In this section, $w = (w_t)_{t \in [0,1]}$ denotes a standard Wiener process starting at w_0 . Then

$$C_t^n = n \int_0^t \sum_j 1_j(s) 4a(w_{t_{j-1}})^2 \left(\int_{t_{j-1}}^s dw_r \right)^2 ds$$

and we see

$$\begin{aligned} C_t^n &= \sum_j \int_{t_{j-1}}^{t_j} 1_{[0,t]}(s) 4a(w_{t_{j-1}})^2 n \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^2 - (s - t_{j-1}) \right\} ds \\ &\quad + \frac{2}{n} \sum_{j: t_j \leq t} a(w_{t_{j-1}})^2 + O_p\left(\frac{1}{n}\right) \\ &\xrightarrow{p} 2 \int_0^t a(w_s)^2 ds = C_t^{\infty}. \end{aligned}$$

In this example, we will consider

$$F_n = \frac{2}{n} \sum_j a(w_{t_{j-1}})^2.$$

Obviously

$$F_n \xrightarrow{p} 2 \int_0^1 a(w_s)^2 ds = C_1^{\infty} \equiv C_{\infty}.$$

This setting is natural in estimation of the diffusion coefficient (volatility), where M^n becomes the “deviation” of the estimator for the cumulative variance of the system and F_n is an estimator of the “asymptotic variance”.

Let $\alpha(x) = a(x)^2$. In order to obtain asymptotic expansion, we will assume the nondegeneracy condition

$$[\mathbf{HW1}] \quad \inf_{x \in \mathbb{R}} |a(x)| > 0.$$

In application to statistical estimation of volatility, this condition is translated as the uniform ellipticity of the diffusion process. For the nondegeneracy of F_{∞} , we need the following condition.

$$[\mathbf{HW2}] \quad \sum_{i=1}^{\infty} |\partial^i \alpha(w_0)| > 0, \text{ where } w_0 \text{ is the initial value of } w.$$

The initial value w_0 can be random but has a compact support.

Set

$$\begin{aligned} \mathfrak{C}_0 &= 2 \int_0^1 a(w_s)^2 ds = C_1^{\infty} \equiv C_{\infty} = F_{\infty}, \\ \mathfrak{C}_1 &= \frac{2}{3} \int_0^1 a(w_s)^3 ds \left(\int_0^1 a(w_s)^2 ds \right)^{-1} \\ \mathfrak{C}_2 &= \int_0^1 a(w_t) \left(\int_t^1 aa'(w_{\nu}) d\nu \right)^2 dt \\ \mathfrak{C}_3 &= \int_0^1 a(w_t) \int_t^1 \{aa'' + (a')^2\}(w_{\nu}) d\nu dt. \end{aligned}$$

Then the random symbol σ is given by

$$\sigma(z, iu, iv) = \mathfrak{C}_1 z(iu)^2 + \mathfrak{C}_2 iu \left(-2u^2 + 4iv \right)^2 + \mathfrak{C}_3 iu \left(-2u^2 + 4iv \right).$$

The asymptotic expansion is done by

$$p_n(z, x) = \phi(z; 0, x)E[\delta_x(\mathfrak{C}_0)] + \frac{1}{\sqrt{n}}\dot{p}(z, x),$$

where

$$\begin{aligned}\dot{p}(z, x) &= E\left[\sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; 0, C_\infty) \delta_x(C_\infty) \right\}\right] \\ &= E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \partial_z^2 \left\{ z \phi(z; 0, x) \right\} \\ &\quad - \partial_z \left(2 \partial_z^2 - 4 \partial_x \right)^2 \left\{ E[\mathfrak{C}_2 \delta_x(\mathfrak{C}_0)] \phi(z; 0, x) \right\} \\ &\quad - \partial_z \left(2 \partial_z^2 - 4 \partial_x \right) \left\{ E[\mathfrak{C}_3 \delta_x(\mathfrak{C}_0)] \phi(z; 0, x) \right\} \\ &= E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \partial_z^2 \left\{ z \phi(z; 0, x) \right\} - 16 \left\{ \partial_x^2 E[\mathfrak{C}_2 \delta_x(\mathfrak{C}_0)] \right\} \partial_z \phi(z; 0, x) \\ &\quad + 4 \left\{ \partial_x E[\mathfrak{C}_3 \delta_x(\mathfrak{C}_0)] \right\} \partial_z \phi(z; 0, x) \\ &=: p_1(z, x) + p_2(z, x) + p_3(z, x).\end{aligned}$$

Remark 7. In order to obtain rough evaluation of the terms involving δ_x , we may apply the IBP formula, the kernel method, or other methods.

Theorem 5. Suppose that [HW1] and [HW2] are satisfied. Then for any positive numbers M and γ ,

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(M_1^n, F_n)] - \int_{\mathbb{R}^2} f(z, x) p_n(z, x) dz dx \right| = o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \rightarrow \infty$, where $\mathcal{E}(M, \gamma)$ is the set of measurable functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying $|f(z, x)| \leq M(1 + |z| + |x|)^\gamma$ for all $z, x \in \mathbb{R}$.

7.2 Proof

7.2.1 Representation of the limit variables

The variable M_1^n admits the representation

$$M_1^n = n^{-\frac{1}{2}} \sum_{j: t_j \leq 1} a(w_{t_{j-1}}) \left((\sqrt{n} \Delta_j w)^2 - 1 \right),$$

where $\Delta_j w = w_{t_j} - w_{t_{j-1}}$. For $\mathring{C}_t^n := \sqrt{n}(C_t^n - C_t^\infty)$, we have

$$\begin{aligned}\mathring{C}_t^n &= \sum_{j: t_j \leq t} \int_{t_{j-1}}^{t_j} 1_{[0, t]}(s) 4n \sqrt{n} a(w_{t_{j-1}})^2 \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^2 - (s - t_{j-1}) \right\} ds \\ &\quad - 2\sqrt{n} \sum_{j: t_j \leq t} \int_{t_{j-1}}^{t_j} \left(a(w_s)^2 - a(w_{t_{j-1}})^2 \right) ds + O_p\left(\frac{1}{\sqrt{n}}\right) \\ &= \sum_{j: t_j \leq t} \int_{t_{j-1}}^{t_j} 1_{[0, t]}(s) 4n \sqrt{n} a(w_{t_{j-1}})^2 \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^2 - (s - t_{j-1}) \right\} ds \\ &\quad - 2\sqrt{n} \sum_{j: t_j \leq t} \int_{t_{j-1}}^{t_j} 2a(w_{t_{j-1}}) a'(w_{t_{j-1}}) (w_s - w_{t_{j-1}}) ds + O_p\left(\frac{1}{\sqrt{n}}\right) \\ &= \sum_{j: t_j \leq t} \int_{t_{j-1}}^{t_j} 1_{[0, t]}(s) 4n \sqrt{n} a(w_{t_{j-1}})^2 \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^2 - (s - t_{j-1}) \right\} ds \\ &\quad + O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Here we know that the supremum of “ $O_p(n^{-1/2})$ ” in $t \in [0, 1]$ is of $O_p(n^{-1/2})$. With the same argument, we have

$$\sqrt{n}(F_n - F_\infty) \xrightarrow{p} 0 = \overset{\circ}{F}_\infty.$$

We should specify the distribution of

$$(M_\infty, \overset{\circ}{C}_\infty)$$

For computations, we will use the discrete filtration $\mathbf{F}^n = (\bar{\mathcal{F}}_t^n)_{t \in [0, 1]}$ with $\bar{\mathcal{F}}_t^n = \mathcal{F}_{[nt]/n}$. The bracket for \mathbf{F}^n is denoted by $\langle\langle \cdot \rangle\rangle$; though it depends on n , we suppress it from notation. Let $H_1(x) = x$ and $H_2(x) = (x^2 - 1)/\sqrt{2}$. Denote $\Delta_j w = w_{t_j} - w_{t_{j-1}}$, which depends on n as well as j . The discrete version of M^n is given by

$$\bar{M}_t^{2,n} = \frac{1}{\sqrt{n}} \sum_{j:t_j \leq t} \sqrt{2}a(w_{t_{j-1}})H_2(\sqrt{n}\Delta_j w).$$

The principal part of $\overset{\circ}{C}^n$ is \mathbf{F}^n -martingale

$$\bar{M}_t^{\xi,n} = \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} 4n\sqrt{n}a(w_{t_{j-1}})^2 \left\{ \left(\int_{t_{j-1}}^s dw_r \right)^2 - (s - t_{j-1}) \right\} ds$$

The discrete version of w is denoted by $\bar{w}_t^n = w_{[nt]/n}$. Then we have

$$\begin{aligned} \langle\langle \bar{w}^n, \bar{w}^n \rangle\rangle_t &= \frac{[nt]}{n} \rightarrow t \\ \langle\langle \bar{M}^{2,n}, \bar{M}^{2,n} \rangle\rangle_t &= \frac{2}{n} \sum_{j:t_j \leq t} a(w_{t_{j-1}})^2 \xrightarrow{ucp} 2 \int_0^t a(w_s)^2 ds \\ \langle\langle \bar{M}^{\xi,n}, \bar{M}^{\xi,n} \rangle\rangle_t &= \frac{16}{3n} \sum_{j:t_j \leq t} a(w_{t_{j-1}})^4 \xrightarrow{ucp} \frac{16}{3} \int_0^t a(w_s)^4 ds \\ \langle\langle \bar{w}^n, \bar{M}^{k,n} \rangle\rangle_t &= 0 \quad (k = 2, \xi) \\ \langle\langle \bar{M}^{2,n}, \bar{M}^{\xi,n} \rangle\rangle_t &= \frac{8}{3n} \sum_{j:t_j \leq t} a(w_{t_{j-1}})^3 \xrightarrow{ucp} \frac{8}{3} \int_0^t a(w_s)^3 ds. \end{aligned}$$

Obviously, the orthogonality between those martingales and any bounded martingales orthogonal to w holds. Therefore, we obtain the following stable convergence with a representation of the limit:

$$(\bar{M}^{2,n}, \bar{M}^{\xi,n}) \xrightarrow{d_s(\mathcal{F})} \left(\int_0^\cdot \sqrt{2}a(w_s)dB_s, \int_0^\cdot \frac{4\sqrt{2}}{3}a(w_s)^2dB_s + \int_0^\cdot \frac{4}{3}a(w_s)^2dB'_s \right),$$

where (B, B') is a two-dimensional standard Wiener process, independent of \mathcal{F} , defined on the extension $\bar{\Omega}$. In particular,

$$(M_\infty, \overset{\circ}{C}_\infty) \stackrel{d}{=} \left(\int_0^1 \sqrt{2}a(w_s)dB_s, \int_0^1 \frac{4\sqrt{2}}{3}a(w_s)^2dB_s + \int_0^1 \frac{4}{3}a(w_s)^2dB'_s \right),$$

so that

$$\check{C}_\infty(\omega, M_\infty) = \mathbb{E}[\overset{\circ}{C}_\infty | \check{\mathcal{F}}] = \frac{4}{3} \int_0^1 a(w_s)^3 ds \left(\int_0^1 a(w_s)^2 ds \right)^{-1} M_\infty.$$

Thus we have

$$\begin{aligned} \check{C}_\infty(z) &= \check{C}_\infty(\omega, z - W_\infty) = \check{C}_\infty(\omega, z) \\ &= \frac{4z}{3} \int_0^1 a(w_s)^3 ds \left(\int_0^1 a(w_s)^2 ds \right)^{-1}. \end{aligned}$$

By definition of the variable in this example, we have $\tilde{F}_\infty(\omega, z) = 0$ and $\tilde{N}_\infty(\omega, z) = 0$. From above computations, the random symbol $\underline{\sigma}(z, iu, iv)$ is given by

$$\underline{\sigma}(z, iu, iv) = \frac{2z}{3} \int_0^1 a(w_s)^3 ds \left(\int_0^1 a(w_s)^2 ds \right)^{-1} (iu)^2.$$

In order to obtain the random symbol $\sigma_{s,r}(iu, iv)$, we note

$$\begin{aligned} D_r C_s^\infty &= 4 \int_r^s aa'(w_\nu) d\nu \mathbf{1}_{\{r \leq s\}}, \\ D_s C_\infty &= D_s C_1^\infty = D_s F_\infty = 4 \int_s^1 aa'(w_\nu) d\nu \mathbf{1}_{\{s \leq 1\}}, \\ D_r D_s C_\infty &= D_r D_s F_\infty = 4 \int_{s \vee r}^1 \{aa'' + (a')^2\}(w_\nu) d\nu. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{s,r} &= 2u^2 \left(-2u^2 + 4iv \right) \int_r^s aa'(w_\nu) d\nu \int_s^1 aa'(w_\nu) d\nu \\ &\quad + \left(-2u^2 + 4iv \right)^2 \int_r^1 aa'(w_\nu) d\nu \int_s^1 aa'(w_\nu) d\nu \\ &\quad + \left(-2u^2 + 4iv \right) \int_{s \vee r}^1 \{aa'' + (a')^2\}(w_\nu) d\nu \end{aligned}$$

and

$$\begin{aligned} \sigma_{t,t}(iu, iv) &= \left(-2u^2 + 4iv \right)^2 \left(\int_t^1 aa'(w_\nu) d\nu \right)^2 \\ &\quad + \left(-2u^2 + 4iv \right) \int_t^1 \{aa'' + (a')^2\}(w_\nu) d\nu. \end{aligned}$$

Thus

$$\begin{aligned} \bar{\sigma}(iu, iv) &= \int_0^1 a(w_t) iu \sigma_{t,t}(iu, iv) dt \\ &= iu \left(-2u^2 + 4iv \right)^2 \int_0^1 a(w_t) \left(\int_t^1 aa'(w_\nu) d\nu \right)^2 dt \\ &\quad + iu \left(-2u^2 + 4iv \right) \int_0^1 a(w_t) \int_t^1 \{aa'' + (a')^2\}(w_\nu) d\nu dt. \end{aligned}$$

Remark 8. From computational point of view, only rough simulation for the conditional expectation of $\bar{\sigma}(iu, iv)$ given F_∞ will be necessary to obtain the second-order term. This is called the *Hybrid I method*.

Thanks to the nondegeneracy of a , it is easy to verify [A1](ii); otherwise, we would be involved in a tedious large deviation argument which we did not want to pursue here. Other conditions in [A1] are also easy to prove.

7.2.2 Nondegeneracy

Here we will briefly discuss the nondegeneracy in Malliavin's sense. We have

$$\begin{aligned} D_r M_t^n &= \sqrt{n} \sum_{j=1}^n 2a(w_{t_{j-1}}) \left(\int_{t_{j-1} \wedge t}^{t_j \wedge t} dw_s \right) \mathbf{1}_{(t_{j-1} \wedge t, t_j \wedge t)}(r) \\ &\quad + \sqrt{n} \sum_{j=1}^n a'(w_{t_{j-1}}) \mathbf{1}_{(0, t_{j-1} \wedge t]}(r) \left(\left(\int_{t_{j-1} \wedge t}^{t_j \wedge t} dw_s \right)^2 - (t_{j-1} \wedge t - t_{j-1} \wedge t) \right) \\ &=: D_1(n, t)_r + D_2(n, t)_r. \end{aligned}$$

Let $\eta_j(t) = \sqrt{n}(w(t_j \wedge t) - w(t_{j-1} \wedge t))$ and

$$\xi_j(t) = n \left((w(t_j \wedge t) - w(t_{j-1} \wedge t))^2 - (t_j \wedge t - t_{j-1} \wedge t) \right).$$

Then

$$\begin{aligned} D_2(n, t)_r &= n^{-\frac{1}{2}} \sum_{j=1}^n a'(w_{t_{j-1}}) 1_{(0, t_{j-1} \wedge t]}(r) \xi_j(t) \\ &= n^{-\frac{1}{2}} \sum_{j=1}^{n-1} \left(\sum_{k=j+1}^n a'(w_{t_{k-1}}) \xi_k(t) \right) 1_{I_j(t)}(r), \end{aligned}$$

where $I_j(t) = (t_{j-1} \wedge t, t_j \wedge t]$. For a while, we assume $t \in \{t_j\}_j$. Hence

$$\sigma_{11}(n, t) := \sigma_{M_t^n} = \frac{1}{n} \sum_{j=1}^n \left[2a(w(t_{j-1})) \eta_j(t) + \frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right]^2,$$

where we read $\sum_{k=n+1}^n \dots = 0$. Moreover,

$$\sigma_{12}(n, t) := \langle M_t^n, F_\infty \rangle_H = \sum_{j=1}^n \left[2a(w_{t_{j-1}}) \eta_j(t) + \frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right] \cdot \int_{I_j(t)} 4 \int_r^1 aa'(w_s) ds dr$$

and

$$\sigma_{22}(t) = \int_0^t \left[4 \int_r^1 aa'(w_s) ds \right]^2 dr.$$

The Malliavin covariance matrix of (M_t^n, F_∞) is then given by

$$\sigma_{(M_t^n, F_\infty)} = \begin{bmatrix} \sigma_{11}(n, t) & \sigma_{12}(n, t) \\ \sigma_{12}(n, t) & \sigma_{22}(t) \end{bmatrix}.$$

Let

$$\sigma(n, t) := \begin{bmatrix} \sigma_{11}(n, t) & \sigma_{12}(n, t) \\ \sigma_{12}(n, t) & \sigma_{22}(t) \end{bmatrix}.$$

Let

$$\tilde{\sigma}_{11}(n, t) = \frac{1}{n} \sum_{j=1}^n \left[2a(w_{t_{j-1}}) \eta_j(t) \right]^2 + \frac{1}{n} \sum_{j=1}^n \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right]^2,$$

$$\tilde{\sigma}_{12}(n, t) = \sum_{j=1}^n \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right] \cdot \int_{I_j(t)} 4 \int_r^1 aa'(w_s) ds dr$$

and

$$\tilde{\sigma}(n, t) = \begin{bmatrix} \tilde{\sigma}_{11}(n, t) & \tilde{\sigma}_{12}(n, t) \\ \tilde{\sigma}_{12}(n, t) & \sigma_{22}(t) \end{bmatrix}.$$

We shall show

$$\|\sigma(n, t) - \tilde{\sigma}(n, t)\|_p = O(n^{-\frac{1}{2}}) \quad (44)$$

for every $p > 1$ and uniformly in t .

Let \mathcal{I} denote the set of sequences $J^{(\nu)} = (J_{n,j}^{(\nu)})$ of multiple Itô stochastic integrals taking the form

$$J_{n,j}^{(\nu)} = n^{\frac{\nu}{2}} \int_{t_{j-1}}^{t_j} dw_{s_1} a_{n,j,1}^{(\nu)}(s_1) \int_{t_{j-1}}^{s_1} dw_{s_2} a_{n,j,2}^{(\nu)}(s_2) \int_{t_{j-1}}^{s_2} \cdots \int_{t_{j-1}}^{s_{\nu}} dw_{s_{\nu}} a_{n,j,\nu}^{(\nu)}(s_{\nu}),$$

where $\{a_{n,j,i}^{(\nu)}; i = 1, \dots, \nu, j = 1, \dots, n, n \in \mathbb{N}\}$ is a family of progressively measurable processes. In the following lemma, $J_{n,j_1}^{(\nu_1)} \cdots J_{n,j_m}^{(\nu_m)}, J_{n,k_1}^{(\mu_1)} \cdots J_{n,k_m}^{(\mu_m)}$ are in \mathcal{I} , and each of them has $a_{n,j,i}^{(*)}$ which may possibly differ from those of other indices ν 's and μ 's even if the values of indices coincide each other.

Lemma 5. *Suppose that*

$$\sup_{\substack{j \in \{1, \dots, n\}, n \in \mathbb{N}, \gamma \in \{0, 1, \dots, m\}, \\ r_1, \dots, r_{\gamma}, s \in [0, 1]}} \|D_{r_1, \dots, r_{\gamma}} \mathcal{O}\|_p < \infty$$

for all $\mathcal{O} = a_{n,j}^{(*)}, b_{n,j}^{(*)}$ and $a_{n,j,i}^{(*)}(s)$, and for every $p > 1$.¹³ Then

(a) Suppose that $b_{n,k}^{(d)}$ are $\mathcal{F}_{t_{k-1}}$ -measurable. Then for $\nu_1, \dots, \nu_m, \mu_1, \dots, \mu_q \in \mathbb{N}$,

$$\frac{1}{n^m} \sum_{j_1, \dots, j_m} E \left[a_{n,j_1}^{(1)} J_{n,j_1}^{(\nu_1)} \cdots a_{n,j_m}^{(m)} J_{n,j_m}^{(\nu_m)} \left(\frac{1}{\sqrt{n}} \sum_{k_1=j_1+1}^n b_{n,k_1}^{(1)} J_{n,k_1}^{(\mu_1)} \right) \cdots \left(\frac{1}{\sqrt{n}} \sum_{k_q=j_m+1}^n b_{n,k_q}^{(q)} J_{n,k_q}^{(\mu_q)} \right) \right] = O\left(\frac{1}{n^{m/2}}\right).$$

(b) For $\nu_1, \dots, \nu_m \in \mathbb{N}$,

$$\frac{1}{n^m} \sum_{j_1, \dots, j_m} E \left[a_{n,j_1}^{(1)} J_{n,j_1}^{(\nu_1)} \cdots a_{n,j_m}^{(m)} J_{n,j_m}^{(\nu_m)} \right] = O\left(\frac{1}{n^{m/2}}\right).$$

The constants in the above estimates depend only on the given supremums.

Proof. First we will show (a). We use the $L^2([0, T])$ -orthogonality between $1_{(t_{j-1}, t_j]}$ and $1_{(t_{k-1}, t_k]}$ for $j \neq k$. If the number of single j_* 's is α , then the outside summation has at most $n^\alpha \times n^{\frac{m-\alpha}{2}}$ terms of such type. The α times IBP-formula for those single j_* 's deduces the order $n^{-m/2}$ if k_1, \dots, k_q are different from any of j_* 's; otherwise, we also get $n^{-1/2}$ in each IBP-formula. Note also that the derivative of b 's do not change the form of "martingale", besides it gives $n^{-1/2}$. After all, total order becomes

$$n^{-m} \times n^\alpha \times n^{\frac{m-\alpha}{2}} \times n^{-\frac{1}{2}\alpha} = n^{-\frac{m}{2}}.$$

In a similar way, we can obtain (b). □

For example, we apply (b) for

$$a_{n,j}^{(c)} = 2a(w_{t_{j-1}}) \times n \int_{I_j(t)} \left(4 \int_r^1 aa'(w_s) ds \right) dr.$$

By Lemma 5, we see

$$\left\| \frac{1}{n} \sum_{j=1}^n 2a(w_{t_{j-1}}) \eta_j(t) \left(\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right) \right\|_p = \left(\frac{1}{\sqrt{n}} \right)$$

and

$$\left\| \sum_{j=1}^n 2a(w_{t_{j-1}}) \eta_j(t) \int_{I_j(t)} 4 \int_r^1 aa'(w_s) ds dr \right\|_p = \left(\frac{1}{\sqrt{n}} \right)$$

¹³ $\gamma = 0$ denotes the case with no derivative.

as $n \rightarrow \infty$ for every $p > 1$. Consequently, we obtain (44).

Now we have

$$\begin{aligned}
\det \tilde{\sigma}(n, t) &= \frac{1}{n} \sum_{j=1}^n \left[2a(w_{t_{j-1}}) \eta_j(t) \right]^2 \int_0^t \left[4 \int_r^1 aa'(w_s) ds \right]^2 dr \\
&\quad + \sum_{j=1}^n \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right]^2 \times \frac{1}{n} \sum_{j=1}^n \int_{I_j(t)} \left[4 \int_r^1 aa'(w_s) ds \right]^2 dr \\
&\quad - \left\{ \sum_{j=1}^n \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(w(t_{k-1})) \xi_k(t) \right] \cdot \int_{I_j(t)} 4 \int_r^1 aa'(w_s) ds dr \right\}^2 \\
&\geq \frac{1}{n} \sum_{j=1}^n \left[2a(w_{t_{j-1}}) \eta_j(t) \right]^2 \int_0^t \left[4 \int_r^1 aa'(w_s) ds \right]^2 dr,
\end{aligned} \tag{45}$$

where we used the Schwarz inequality as well as $|I_j(t)| \leq 1/n$.

Let $t_n = 1/2$ and $c_0 = \inf_{x \in \mathbb{R}} |a(x)|$. Define s_n by

$$s_0 := s_n = \frac{1}{2} c_0^2 \int_0^{\frac{1}{2}} \left[4 \int_r^1 aa'(w_s) ds \right]^2 dr.$$

In particular, s_n does not depend on n in this case.

We consider the system of stochastic differential equations:

$$\begin{cases} dw_t &= dw_t, \\ df_t &= 2\alpha(w_t) dt. \end{cases}$$

For this system, we have $V_0 = 2\alpha(w)\partial_2$, $V_1 = \partial_1$, where ∂_1 and ∂_2 correspond to w and f , respectively. We see that Condition [HW2] together with [HW1] applied at $w(0)$ implies the Hörmander condition for this system; see for example Ikeda and Watanabe. The boundedness of derivatives of the coefficients assumed there can be removed in our case by means of a large deviation argument. In particular, $F_\infty = f_1$ is nondegenerate, even up to $t = 1/2$, that is,

$$s_0^{-1} \in L^{\infty-}. \tag{46}$$

Since M_t^n and F_n are asymptotically orthogonal in the H -space in the sense of (45), the convergence is sufficiently fast as shown by (44), and s_0 is nondegenerate as (46), we conclude the uniform nondegeneracy of (M_t^n, F_∞) for $t \geq 1/2$, as follows. Due to $\sigma_{(M_t^n, F_\infty)} \geq \sigma(n, t)$ by definition, for every $K > 0$ and $t_n^\dagger = \min\{t_j; t_j \geq 1/2\}$,

$$\begin{aligned}
\sup_{t \geq 1/2} P \left[\det \sigma_{(M_t^n, F_\infty)} < s_0 \right] &\leq P \left[\det \sigma(n, t_n^\dagger) < 1.5s_0 \right] + O(n^{-K}) \\
&\leq P \left[\det \tilde{\sigma}(n, t_n^\dagger) < 2s_0 \right] + O(n^{-K}) \\
&= O(n^{-K})
\end{aligned}$$

as $n \rightarrow \infty$.

7.2.3 Proof of Theorem 5

Theorem 5 now follows from the results of the preceding subsections.

7.3 Studentization

We shall consider the expansion of the expectation $E[g(F_n^{-\frac{1}{2}} M_1^n)]$. This form corresponds to a studentized statistic in the statistical context. The contribution of the principal part is given by

$$\int g\left(\frac{z}{\sqrt{x}}\right) \phi(z; 0, x) E[\delta_x(\mathfrak{C}_0)] dz dx = \int g(z) \phi(z; 0, 1) dz \int p^{C_\infty}(x) dx = \int g(z) \phi(z; 0, 1) dz$$

Let $g \in \mathcal{S}(\mathbb{R})$. For the second-order terms, we have

$$\begin{aligned}
\int g\left(\frac{z}{\sqrt{x}}\right) p_1(z, x) dz dx &= \int g\left(\frac{z}{\sqrt{x}}\right) E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \partial_z^2 \left\{ z \phi(z; 0, x) \right\} dz dx \\
&= \int \frac{1}{x} g''\left(\frac{z}{\sqrt{x}}\right) E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] z \phi(z; 0, x) dz dx \\
&= \int g''(z) E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \frac{z}{\sqrt{x}} \phi(z; 0, 1) dz dx \\
&= \int E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \frac{1}{\sqrt{x}} dx \cdot \int g(z) (z^3 - 3z) \phi(z; 0, 1) dz \\
&= E\left[\frac{\mathfrak{C}_1}{\sqrt{\mathfrak{C}_0}}\right] \cdot \int g(z) (z^3 - 3z) \phi(z; 0, 1) dz
\end{aligned}$$

Here we used

$$\begin{aligned}
\int E[\mathfrak{C}_1 \delta_x(\mathfrak{C}_0)] \frac{1}{\sqrt{x}} dx &= \int E\left[\frac{\mathfrak{C}_1}{\sqrt{\mathfrak{C}_0}} \delta_x(\mathfrak{C}_0)\right] dx \\
&= - \int \partial_x E\left[\frac{\mathfrak{C}_1}{\sqrt{\mathfrak{C}_0}} 1_{\mathbb{R}_+}(\mathfrak{C}_0 - x)\right] dx \\
&= E\left[\frac{\mathfrak{C}_1}{\sqrt{\mathfrak{C}_0}}\right]
\end{aligned}$$

Define polynomials $P_{\beta, \nu}(z, x)$ by

$$(-\partial_x)^\beta g\left(\frac{z}{\sqrt{x}}\right) = \sum_{\nu \leq \beta} P_{\beta, \nu}\left(\frac{z}{\sqrt{x}}, \frac{1}{\sqrt{x}}\right) g^{(\nu)}\left(\frac{z}{\sqrt{x}}\right)$$

for a β -times differentiable function g . Set

$$Q_{\alpha, \beta, \nu}(z, x) = x^\alpha P_{\beta, \nu}(z, x).$$

For $g \in \mathcal{S}(\mathbb{R})$ and any smooth functional \mathfrak{D} , we have

$$\begin{aligned}
&\int g\left(\frac{z}{\sqrt{x}}\right) \partial_z^\alpha \partial_x^\beta \left\{ E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] \phi(z; 0, x) \right\} dz dx \\
&= \int g\left(\frac{z}{\sqrt{x}}\right) \partial_x^\beta \left\{ E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] x^{-\frac{1}{2}(1+\alpha)} \partial_y^\alpha \phi(y; 0, 1) \Big|_{y=zx^{-1/2}} \right\} dz dx \\
&= \int \sum_{\nu \leq \beta} P_{\beta, \nu}\left(\frac{z}{\sqrt{x}}, \frac{1}{\sqrt{x}}\right) g^{(\nu)}\left(\frac{z}{\sqrt{x}}\right) \left\{ E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] x^{-\frac{1}{2}(1+\alpha)} \partial_y^\alpha \phi(y; 0, 1) \Big|_{y=zx^{-1/2}} \right\} dz dx \\
&= \int \sum_{\nu \leq \beta} P_{\beta, \nu}\left(y, \frac{1}{\sqrt{x}}\right) g^{(\nu)}(y) E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] x^{-\frac{1}{2}\alpha} \partial_y^\alpha \phi(y; 0, 1) dy dx \\
&= \int g(y) \sum_{\nu \leq \beta} (-\partial_y)^{(\nu)} \left\{ \partial_y^\alpha \phi(y; 0, 1) \int P_{\beta, \nu}\left(y, \frac{1}{\sqrt{x}}\right) E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] x^{-\frac{1}{2}\alpha} dx \right\} dy \\
&= \int g(y) \sum_{\nu \leq \beta} (-\partial_y)^{(\nu)} \left\{ \partial_y^\alpha \phi(y; 0, 1) \int E\left[Q_{\alpha, \beta, \nu}\left(y, \frac{1}{\sqrt{\mathfrak{C}_0}}\right) \mathfrak{D} \delta_x(\mathfrak{C}_0)\right] dx \right\} dy,
\end{aligned}$$

therefore we obtain the formula

$$\begin{aligned}
&\int g\left(\frac{z}{\sqrt{x}}\right) \partial_z^\alpha \partial_x^\beta \left\{ E[\mathfrak{D} \delta_x(\mathfrak{C}_0)] \phi(z; 0, x) \right\} dz dx \\
&= \int g(y) \sum_{\nu \leq \beta} (-\partial_y)^{(\nu)} \left\{ \partial_y^\alpha \phi(y; 0, 1) E\left[Q_{\alpha, \beta, \nu}\left(y, \frac{1}{\sqrt{\mathfrak{C}_0}}\right) \mathfrak{D}\right] \right\} dy.
\end{aligned}$$

By definition,

$$\begin{aligned} Q_{\alpha,0,0}(y,x) &= x^\alpha, \\ Q_{\alpha,1,0}(y,x) &= 0, \quad Q_{\alpha,1,1}(y,x) = \frac{1}{2}yx^{\alpha+2} \\ Q_{\alpha,2,0}(y,x) &= 0, \quad Q_{\alpha,2,1}(y,x) = \frac{3}{4}yx^{\alpha+4}, \quad Q_{\alpha,2,2}(y,x) = \frac{1}{4}y^2x^{\alpha+4}. \end{aligned}$$

Applying the formulas, we have

$$\begin{aligned} \int g\left(\frac{z}{\sqrt{x}}\right)p_2(z,x) dzdx &= - \int g\left(\frac{z}{\sqrt{x}}\right)\partial_z\left(2\partial_z^2 - 4\partial_x\right)^2 \left\{E[\mathfrak{C}_2\delta_x(\mathfrak{C}_0)]\phi(z;0,x)\right\} dzdx \\ &= \int g(y) \left[-4\partial_y^5\phi(y;0,1)E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2] \right. \\ &\quad -8\partial_y\left(y\partial_y^3\phi(y;0,1)\right)E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2] \\ &\quad +12\partial_y\left(y\partial_y\phi(y;0,1)E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2]\right) \\ &\quad \left. -4\partial_y^2\left(y^2\partial_y\phi(y;0,1)E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2]\right) \right] dy \\ &= \int g(y)E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2](12y)\phi(y;0,1) dy \end{aligned}$$

and

$$\begin{aligned} \int g\left(\frac{z}{\sqrt{x}}\right)p_3(z,x) dzdx &= - \int g\left(\frac{z}{\sqrt{x}}\right)\partial_z\left(2\partial_z^2 - 4\partial_x\right)\left\{E[\mathfrak{C}_3\delta_x(\mathfrak{C}_0)]\phi(z;0,x)\right\} dzdx \\ &= \int g(y) \left[-2\partial_y^3\phi(y;0,1)E[\mathfrak{C}_0^{-\frac{3}{2}}\mathfrak{C}_3] - 2\partial_y\left(y\partial_y\phi(y;0,1)E[\mathfrak{C}_0^{-\frac{3}{2}}\mathfrak{C}_3]\right) \right] dy \\ &= \int g(y)E[\mathfrak{C}_0^{-\frac{3}{2}}\mathfrak{C}_3](-2y)\phi(y;0,1) dy. \end{aligned}$$

After all, we obtain

$$\begin{aligned} q_n(z) &= \phi(z;0,1) + \frac{1}{\sqrt{n}} \left\{ E[\mathfrak{C}_0^{-\frac{1}{2}}\mathfrak{C}_1](z^3 - 3z) \right. \\ &\quad + E[\mathfrak{C}_0^{-\frac{5}{2}}\mathfrak{C}_2](12z) \\ &\quad \left. + E[\mathfrak{C}_0^{-\frac{3}{2}}\mathfrak{C}_3](-2z) \right\} \phi(z;0,1) \end{aligned}$$

as the second-order approximate density to the distribution of $F_n^{-1/2}M_1^n$.

8 Quadratic form of a diffusion process

We shall apply and extend the result in Section 7 to the quadratic variation of a diffusion process. It is called a realized volatility in financial context recently. We consider a diffusion process satisfying the Itô integral equation (1). Here b and σ are assumed to be smooth with bounded derivatives of positive order.

For simplicity we only treat the one-dimensional case; multivariate analogue is straightforward. Even extension to Itô processes is also possible but the descriptions would be involved. We write b_t for $b(X_t)$ and σ_t for $\sigma(X_t)$. The Itô decomposition of $\sigma_t = \sigma(X_t)$ is denoted by

$$\sigma_t = \sigma_0 + \int_0^t \sigma_s^{[1]} dw_s + \int_0^t \sigma_s^{[0]} ds.$$

Though $\sigma_s^{[1]}$ and $\sigma_s^{[0]}$ have a simple expression with b , σ and X_s , those symbols are convenient to simplify the notation. This rule will be applied for other functionals.

We consider the quadratic form (2) of the increments of X with strongly predictable kernel. Here we are interested in the asymptotic expansion of the normalized error

$$Z_n = \sqrt{n}(U_n - U_\infty)$$

for U_∞ in (3). We are assuming $c \in C_+^\infty(\mathbb{R})$.

8.1 Stochastic expansion

We will need a stochastic expansion of Z_n .

Lemma 6. Z_n admits the following stochastic expansion:

$$Z_n = M_1^n + \frac{1}{\sqrt{n}}N_n,$$

where

$$M_t^n = \sqrt{n} \sum_{j=1}^n 2c_{t_{j-1}} \sigma_{t_{j-1}}^2 \int_{t_{j-1} \wedge t}^{t_j \wedge t} \int_{t_{j-1}}^s dw_r dw_s,$$

and

$$\begin{aligned} N_n = & 6n \sum_{j=1}^n c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^s dw_u dw_s dw_t \\ & + 2 \sum_{j=1}^n c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_j} dw_t + 2n \sum_{j=1}^n c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) dw_t \\ & + n^{-1} \sum_{j=1}^n c_{t_{j-1}} b_{t_{j-1}}^2 + n^{-1} \sum_{j=1}^n c_{t_{j-1}} \sigma_{t_{j-1}} b_{t_{j-1}}^{[1]} \\ & - n \sum_{j=1}^n c_{t_{j-1}}^{[1]} \sigma_{t_{j-1}}^2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt \\ & - \frac{1}{2n} \sum_{j=1}^n c_{t_{j-1}}^{[0]} \sigma_{t_{j-1}}^2 - \frac{1}{n} \sum_{j=1}^n c_{t_{j-1}}^{[1]} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} + o_M(1). \end{aligned}$$

Here $o_M(1)$ denotes a term of $o(1)$ as $n \rightarrow \infty$ with respect to $\mathbb{D}_{s,p}$ -norms of any order. The families $\{M_t^n\}_{t \in [0,1], n \in \mathbb{N}}$ and $\{N_n\}_{n \in \mathbb{N}}$ are bounded in every $\mathbb{D}_{s,p}$ -norm.

By somewhat long computations, it is possible to obtain the above lemma. We omit details.

8.2 Asymptotic expansion

For a reference variable, we will consider

$$F_n = \frac{1}{n} \sum_{j=1}^n \beta(X_{t_{j-1}}) \quad \text{or} \quad F_n = F_\infty := \int_0^1 \beta(X_t) dt,$$

where $\beta \in C_+^\infty(\mathbb{R}, \mathbb{R}^{d_1})$. The results will be the same in these cases. It is statistically natural to consider those functionals because, for example, F_∞ gives the conditional asymptotic variance of the estimation error Z_1^n as in Section 7. We will derive asymptotic expansion of the joint distribution of (M_1^n, F_n) .

Let $a(x) = c(x)\sigma(x)^2$. Let

$$V_0(x_1, x_2) = \begin{bmatrix} b(x_1) - \frac{1}{2}\sigma(x_1)\partial_{x_1}\sigma(x_1) \\ \beta(x_1) \end{bmatrix} \quad \text{and} \quad V_1(x_1, x_2) = \begin{bmatrix} \sigma(x_1) \\ 0 \end{bmatrix}$$

for $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{d_1}$. The Lie algebra generated by

$$V_1, [V_i, V_j] \ (i, j = 0, 1), [V_i, [V_j, V_k]] \ (i, j, k = 0, 1), \dots$$

at (x_1, x_2) is denoted by $\text{Lie}[V_0; V_1](x_1, x_2)$.

Assume that $\text{supp}(X_0)$ is compact. Moreover, for nondegeneracy, we assume

$$[\mathbf{H1}] \quad \inf_{x \in \mathbb{R}} |a(x)| > 0.$$

$$[\mathbf{H2}] \quad \text{Lie}[V_0; V_1](X_0, 0) = \mathbb{R}^{1+d_1} \text{ a.s.}$$

Remark 9. Under [H1], both $\text{ess. inf } |\sigma(X_0)| > 0$ and $\text{ess. inf } |c(X_0)| > 0$. Then [H2] is equivalent to the linear hull $L[\partial_{x_1}^i \beta(X_0); i \in \mathbb{N}] = \mathbb{R}^{d_1}$ a.s. It is rather simple but we prefer to keep [H2], which is suitable for more general form of F_∞ .

Remark 10. Condition [H1] is usually from the uniform ellipticity of the diffusion process X_t and a reasonable choice of the estimator for the quadratic variation. In this sense, it is a natural assumption in statistical context.

Remark 11. Consider a $(1 + d_1)$ -dimensional stochastic integral equation

$$\check{X}_t = \check{X}_0 + \int_0^t V_0(\check{X}_s) ds + \int_0^t V_1(\check{X}_s) \circ dw_s, \quad t \in [0, 1].$$

Then $\check{X}_1 = (X_1, F_\infty)$, and the nondegeneracy condition entails the nondegeneracy of F_∞ in particular.

We see $\tilde{W}_\infty(z) = 0$ and $\tilde{F}_\infty(z) = 0$. It is necessary to specify the limit $(M_\infty, \mathring{C}_\infty, N_\infty)$. The “martingale part” of N_n with respect to \mathbf{F}^n is given by

$$\begin{aligned} \dot{N}_t^n &= 6n \sum_{j:t_j \leq t} c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^s dw_u dw_s dw_t \\ &\quad + 2 \sum_{j:t_j \leq t} c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_j} dw_t + 2n \sum_{j:t_j \leq t} c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) dw_t \\ &\quad - n \sum_{j:t_j \leq t} c_{t_{j-1}}^{[1]} \sigma_{t_{j-1}}^2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt. \end{aligned}$$

We redefine $\bar{M}^{2,n}$ and $\bar{M}^{\xi,n}$ by the same equations in Section 7.2.1 but with $a(X_{t_{j-1}})$ in place of $a(w_{t_{j-1}})$. Then

$$\begin{aligned} \langle\langle \bar{w}^n, \bar{w}^n \rangle\rangle_t &= \frac{[nt]}{n} \rightarrow t \\ \langle\langle \bar{M}^{2,n}, \bar{M}^{2,n} \rangle\rangle_t &= \frac{2}{n} \sum_{j:t_{j-1} \leq t} a(X_{t_{j-1}})^2 \rightarrow^p 2 \int_0^t a(X_s)^2 ds \\ \langle\langle \bar{M}^{\xi,n}, \bar{M}^{\xi,n} \rangle\rangle_t &= \frac{16}{3n} \sum_{j:t_{j-1} \leq t} a(X_{t_{j-1}})^4 \rightarrow^p \frac{16}{3} \int_0^t a(X_s)^4 ds \\ \langle\langle \bar{w}^n, \dot{N}^n \rangle\rangle_t &\rightarrow^p \int_0^t k_s ds \\ \langle\langle \bar{w}^n, \bar{M}^{k,n} \rangle\rangle_t &= 0 \quad (k = 2, \xi) \\ \langle\langle \bar{M}^{2,n}, \bar{M}^{\xi,n} \rangle\rangle_t &= \frac{8}{3n} \sum_{j:t_{j-1} \leq t} a(X_{t_{j-1}})^3 \rightarrow^p \frac{8}{3} \int_0^t a(X_s)^3 ds, \\ \langle\langle \bar{M}^{2,n}, \dot{N}^n \rangle\rangle_t &\rightarrow^p 0, \\ \langle\langle \bar{M}^{\xi,n}, \dot{N}^n \rangle\rangle_t &\rightarrow^p 0, \\ \langle\langle \dot{N}^n, \dot{N}^n \rangle\rangle_t &\rightarrow^p \int_0^t q_s^2 ds \end{aligned}$$

as $n \rightarrow \infty$ for each $t \in [0, 1]$, where \mathbb{R}_+ -valued process q_t takes the form

$$q_t^2 = p(c_t, c_t^{[1]}, b_t, \sigma_t, \sigma_t^{[1]})$$

for some polynomial p ; it is possible to give an explicit expression of p , however we do not need the precise form of q_t later. The orthogonality of $\bar{M}^{2,n}$, $\bar{M}^{\xi,n}$ and \dot{N}^n to any bounded martingale orthogonal to w is obvious, thus with a representation of \dot{C}_t^n in Section 7.2.1 with $a(X_{t_{j-1}})$ for $a(w_{t_{j-1}})$, and those of M_t^n and N_n , we obtain

$$\begin{aligned} (M_\infty, \dot{C}_\infty, N_\infty) &=^d \left(\int_0^1 \sqrt{2}a(X_s)dB_s, \int_0^1 \frac{4\sqrt{2}}{3}a(X_s)^2dB_s + \int_0^1 \frac{4}{3}a(X_s)^2dB'_s, \right. \\ &\quad \left. \int_0^1 k_s dw_s + \int_0^1 \sqrt{q_s^2 - k_s^2} dB_s'' + \int_0^1 h_s ds \right), \end{aligned}$$

where (B, B', B'') is a three-dimensional standard Wiener process, independent of \mathcal{F} , defined on the extension $\tilde{\Omega}$, and

$$h_t = c_t b_t^2 + c_t b_t^{[1]} \sigma_t - \frac{1}{2} c_t^{[0]} \sigma_t^2 - c_t^{[1]} \sigma_t \sigma_t^{[1]}.$$

and

$$k_t = 2c_t b_t \sigma_t + c_t \sigma_t \sigma_t^{[1]} - \frac{1}{2} c_t^{[1]} \sigma_t^2.$$

Since

$$\tilde{N}_\infty(z) = \int_0^1 k_t dw_t + \int_0^1 h_t dt,$$

the random symbol $\underline{\sigma}(z, iu, iv)$ is given by

$$\underline{\sigma}(z, iu, iv) = \frac{2z}{3} \int_0^1 a(X_s)^3 ds \left(\int_0^1 a(X_s)^2 ds \right)^{-1} (iu)^2 + iu \int_0^1 k_t dw_t + iu \int_0^1 h_t dt.$$

Let us find the anticipative random symbol $\bar{\sigma}(iu, iv)$. Recall that $\alpha(x) = a(x)^2$,

$$C_s^\infty = 2 \int_0^s \alpha(X_t) dt, \quad C_\infty = 2 \int_0^1 \alpha(X_t) dt, \quad F_\infty = \int_0^1 \beta(X_t) dt \quad \text{and} \quad W_\infty = 0.$$

The random symbol $\sigma_{s,r}(iu, iv)$ admits the expression

$$\begin{aligned} &\sigma_{s,r}(iu, iv) \\ &= u^2 \int_r^s \alpha'(X_t) D_r X_t dt \left(-u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t) [v] D_s X_t dt \right) \\ &\quad + \left(-u^2 \int_r^1 \alpha'(X_t) D_r X_t dt + i \int_r^1 \beta'(X_t) [v] D_r X_t dt \right) \left(-u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t) [v] D_s X_t dt \right) \\ &\quad + \left(-u^2 \int_{r \vee s}^1 \{ \alpha''(X_t) D_r X_t D_s X_t + \alpha'(X_t) D_r D_s X_t \} dt + i \int_{r \vee s}^1 \{ \beta''(X_t) [v] D_r X_t D_s X_t + \beta'(X_t) [v] D_r D_s X_t \} dt \right) \end{aligned}$$

for $r \leq s$, where the prime $'$ stands for the derivative in $x_1 \in \mathbb{R}$. The processes $D_s X_t$ and $D_r D_s X_t$ are determined according to routine; $D_s X_t$ satisfies the equation

$$D_s X_t = \sigma(X_s) + \int_s^t b'(X_{t_1}) D_s X_{t_1} dt_1 + \int_s^t \sigma'(X_{t_1}) D_s X_{t_1} dw_{t_1}$$

for $s \leq t$, and

$$\begin{aligned} D_r D_s X_t &= \sigma'(X_s) D_r X_s + \int_s^t b''(X_{t_1}) D_r X_{t_1} D_s X_{t_1} dt_1 + \int_s^t b'(X_{t_1}) D_r D_s X_{t_1} dt_1 \\ &\quad + \int_s^t \sigma''(X_{t_1}) D_r X_{t_1} D_s X_{t_1} dw_{t_1} + \int_s^t \sigma'(X_{t_1}) D_r D_s X_{t_1} dw_{t_1} \end{aligned}$$

for $r < s \leq t$. Those equations form a graded system of partially linear equations; therefore the L^p -estimates of the solution are at hand. Now we obtain the anticipative random symbol

$$\bar{\sigma}(iu, iv) = \int_0^1 iu a(X_s) \sigma_{s,s}(iu, iv) ds$$

with

$$\begin{aligned} \sigma_{s,s}(iu, iv) &= \left(-u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t) [v] D_s X_t dt \right)^2 \\ &\quad - u^2 \int_s^1 \{ \alpha''(X_t) (D_s X_t)^2 + \alpha'(X_t) D_s D_s X_t \} dt \\ &\quad + i \int_s^1 \{ \beta''(X_t) [v] (D_s X_t)^2 + \beta'(X_t) [v] D_s D_s X_t \} dt \end{aligned}$$

As before, define the total random symbol σ by (10) and the density function $p_n(z, x) \in C^\infty(\mathbb{R}^{1+d_1})$ by (11). By a quite similar argument as we proved Theorem 5, it is easy to obtain the following theorem. See [32] for details.

Theorem 6. *Suppose that [H1] and [H2] are satisfied. Then for any positive numbers M and γ ,*

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(Z_n, F_n)] - \int_{\mathbb{R}^{1+d_1}} f(z, x) p_n(z, x) dz dx \right| = o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \rightarrow \infty$, where $\mathcal{E}(M, \gamma)$ is the set of measurable functions $f : \mathbb{R}^{1+d_1} \rightarrow \mathbb{R}$ satisfying $|f(z, x)| \leq M(1 + |z| + |x|)^\gamma$ for all $(z, x) \in \mathbb{R} \times \mathbb{R}^{d_1}$.

Remark 12. The hybrid I method with a rough Monte-Carlo method in the second order term is useful in the application of the expansion formula to numerical approximation. Obviously, the asymptotic expansion applies to statistical hypothesis testing. We will show its applications to prediction and option pricing in other papers.

Remark 13. While our method working in the mixed normal limit case enabled us to introduce a *conditioning* variable as F_n , it is possible to consider versions of our results without F_n . It will reduce the regularity condition of smoothness, that is, indices of differentiability of other variables. Of course, in that case, we only obtain a single (not joint) expansion.

Remark 14. Conditional limit theorems are in our scope. Indeed, it is also possible to obtain asymptotic expansion of the conditional distribution, as it was reported at the meetings mentioned in the footnote of p.1.

Remark 15. In [A2^b] and [A2], the index of differentiability of N_n etc. could be reduced by more sophisticated multiple truncations.

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